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THE ARIES PROGRAM: COORDINATES, TRANSFORMATIONS, TRAJECTORIES AND TRACKING

Bliss L. Diamond, et al

Massachusetts Institute of Technology

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MASSACHUSETTS INSTITUTE OF TECHNOLOGY

LINCOLN LABORATORY

THE ARIES PROGRAM: COORDINATES, TRANSFORMATIONS, TRAJECTORIES AND TRACKING

B. L. DIAMOND

Group 37

M. E. AUSTIN

KREMS

TECHNICAL NOTE 1975-15

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LEXINGTON MASSACHUSETTS

ABSTRACT

This report presents the relations necessary to define the motion of a target in the gravitational field of the earth. In order to express this target motion in the frame of a radar, an appropriate set of coordinate systems (and transformations) is introduced. Target tracking in the form of a Maximum Likelihood Estimator is discussed. The problem of interceptor miss distance is treated from the standpoint of the uncertainty volumes associated with estimated target state vectors.

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SECTION 1--INTRODUCTION

ARIES is a system simulation computer program developed by Lincoln Laboratory to study radar tracking and command guided intercepts in a realistic radar environment. Written in FORTRAN and designed for execution on the CDC 6600 computer, it has considerable versatility in the specification of radar, target, tracking and environmental models.

1.1 Program Purpose

ARIES is designed to be a useful analytical tool for several allied areas. Originally ARIES was used in the strategic BMD area to estimate the metric state vector (position and velocity) of tracked targets, and then to extrapolate ahead in time to determine an intercept point for an ICBM. The radar measurements were subject to environmental effects which were reflected in the intercept miss distances. Refraction and scintillation models were used in ARIES, and the effects of various calibration schemes on target location accuracies were studied. ARIES was also used to study the problem of multipath in low angle tracking and to examine the effectiveness of various proposed schemes to overcome degradations in prediction caused by multipath. The use of ARIES in BMD studies was terminated in the summer of 1974.

More recently, a modified endoatmospheric version of ARIES, known as the HWLPTR (Hostile Weapons Location Projectile Tracking Radar) Program, has been used in the tactical area for the evaluation of a radar's performance in "backtracking" an incoming artillery or mortar shell to determine its point of launch. The resulting point estimation error CEP values also assist in the evaluation of the drag model error effects and in the study of the overall performance of hostile weapon location systems.

1.2 Program Features

Since the ARIES Program provides a fairly elaborate system simulation, it is useful to tabulate the various features incorporated in ARIES. The major components of the simulation are summarized below:

1. Target trajectories---accepts input of target state vectors in several coordinate systems. Trajectories with launch angle, reentry angle or minimum energy constraints from a given launch location to an impact point are also available.
2. Radar models---both mechanically steered (dish) and phased array radars are modeled. Radar sensitivity, beamwidth, frequency and location are specified by inputs. Range and angle measurement precision are also specified by inputs.
3. Radar measurement modeling
   a. Target modeling---static cross-section measurements on real targets are used in conjunction with rigid body dynamics (Euler's

equations) to obtain realistic dynamic RCS simulations. Constant and sinusoidal, as well as an analytic cylinder, RCS options are also available.

b. Noise and propagation effects---radar measurements are corrupted by receiver noise (S/N dependent), range-independent noise effects (jitter, quantization, etc.) and uncorrected propagation effects. Tropospheric and ionospheric refraction are assumed to be corrected to within random percentages (input parameters) of the true values. Ionospheric scintillation and multipath effects corrupt the data but are assumed to be uncorrected in the measurement model.

4. Trajectory estimation (Target Tracking)---Maximum Likelihood Estimation (MLE) of the target trajectory is performed based on measurement data collected at specified PRF's over specified track intervals. Individual measurements are weighted according to their measurement variances. ARIES could be easily extended to use recursive tracking algorithms.

5. Target Discrimination---(Not presently implemented.) Conceptually, discrimination algorithms would be implemented to determine whether a particular simulated target constituted a threat to the defended area.

6. Interceptor modeling---(Not presently implemented.) Flight characteristics of one or more interceptor types would be utilized to

conduct a command guided intercept. Currently, the program extrapolates the estimated and true target state vectors to various time (or altitude) points after termination of track to obtain miss distances. Miss distance statistics are computed from the accumulated miss distances observed on a series of Monte Carlo tests.

In addition to the above simulation components, ARIES also accommodates multiple radars, multiple targets and multiple track intervals on a given simulation run. The feature of making many Monte Carlo runs for a given scenario permits the generation of meaningful miss distance statistics. A building block/subroutine program structure lends itself to reasonable straightforward modifications of or additions to the program.

The input/output of the ARIES Program is engineer oriented. For input, simulation data cards are conveniently grouped into "packets" (each packet defines a target model, a radar model, an environmental model, etc.) which the engineer may simply stack up, together with packets specifying the desired simulation "scenario". For output, an 8½" x 11" ARIES Test Report (see Appendix II of Reference 1) is generated which provides the engineer with descriptions of his input model parameters and scenarios, along with the resultant simulation data and statistics. The outputs are all organized into logical sections which are indexed for ready reference. Outputs from ARIES also include trajectory plots superimposed on a world map, plots of true and measured target cross-section, and a radar measurements tape containing metric and RCS data for processing by other programs.

1.3 Program Documentation

The ARIES Program is documented in three separate Lincoln Laboratory Technical Notes as follows:

1. The ARIES Program - A General Overview and Users' Guide
2. The ARIES Program - Coordinates, Transformations, Trajectories and Tracking
3. The ARIES Program - Analysis and Generation of Simulated Radar Measurements

The first report presents a general discussion of the ARIES Program, including the logical organization of the program and descriptions of all subroutines. All of the options available to a user are discussed and the methods of setting up the input "packets", including controls to activate the various options, are presented. A typical run, including a complete ARIES Test Report (output), is discussed in this first volume.

The second and third volumes contain all of the relevant mathematics and the models used in ARIES. Most of the deterministic mathematics (coordinate systems and transformations, trajectory generation and estimation, miss distance calculations, etc.) are in the second report. The third report is primarily concerned withthe generation of radar measurements, including the corruptive effects of noise, radar biases, propagation and time-varying radar cross-section.

1.4 Organization of This Report

In Section 2, some aspects of the WGS66 earth model are presented. This includes a truncated version of the series representation of the gravitational field. The various coordinate systems used in the ARIES Program are defined in Section 3. Target accelerations due to gravitational forces and due to atmospheric drag forces (not implemented in ARIES) are derived in Section 4. A simple predictor-corrector algorithm for the integration of the target equations of motion is also derived. In Section 5, all of the transformations among the various coordinate systems are derived. The problem of defining realistic target trajectories is addressed in Section 6. Only targets which are launched from and impact on the earth are considered. For given launch and impact points, the user can also specify the launch elevation angle or the reentry angle or he can request a minimum energy trajectory. The trajectories are first derived for a central force field (Keplerian motion) and then perturbed to account for the actual gravitational field. Section 7 presents one type of target tracking algorithm; namely, an "after-the-fact" maximum likelihood estimate of the trajectory which best fits the accumulated radar measurements. From the state vectors generated by trajectory fitting on several Monte Carlo runs and the known (true) target position, one can develop an error covariance matrix representing the uncertainty in predicted target position at an assumed intercept point. This covariance matrix and the related handover error ellipsoid and error sphere are presented in Section 8.

SECTION 2--EARTH MODEL

The model for the earth used in the ARIES Program is taken directly from the 1966 World Geodetic Survey (WGS-66). In this model the earth's shape is given as a surface of revolution obtained by rotating an ellipse around its minor axis. The resulting surface is referred to as an ellipsoid or as an oblate spheroid. As part of this same survey, the coefficients required for an expansion of the earth gravitational field in spherical harmonics were also derived from the measured data.

The parameters of the WGS-66 earth model, as used in the ARIES Program, will be defined and summarized in the following sections.

2.1 Earth Ellipsoid

The representation of the earth as an ellipsoid is principally a matter of mathematical convenience; that is, actual points on the earth's surface will depart in varying amounts from the corresponding points on the ellipsoid. However, on average, the ellipsoid will be a good representation. The earth ellipsoid is defined by the Equatorial and North polar radii (semi-major and semi-minor axes). These values are:

$$R_e = 6378.145 \text{ Km} \tag{2-1}$$

$$R_n = 6356.760 \text{ Km} \tag{2-2}$$

Two other parameters of the ellipse, the flattening factor f and the eccentricity ε, are also of interest

$$f = \frac{R_e - R_n}{R_e} = \frac{1}{298.25} \quad (2\text{-}3)$$

$$\varepsilon = \sqrt{1 - \left(\frac{R_n}{R_e}\right)^2} = \sqrt{f(2-f)} = .081820 \quad (2\text{-}4)$$

The oblique view of the earth given in Figure 2.1 and the cross-sectional view given in Figure 2.2 aid in the definition of positions on the earth surface. Longitude is defined as the angle between the plane containing the Greenwich meridian and a plane containing any other meridian; longitude is measured in an easterly direction from the Greenwich meridian.

The earth rotates around its polar axis, as indicated in Figure 2.1, with an angular rate of

$$\omega_e = 7.29211515 \times 10^{-5} \text{ radians/second} \quad (2\text{-}5)$$

There are two latitude angles defined for the earth ellipsoid, as indicated in Figure 2.2. Both angles are referenced to the equatorial plane. The geodetic latitude ϕ is defined as the angle between the local normal to the earth ellipsoid and the equatorial plane. Mathematically, it is defined by

$$\tan\phi = -\frac{1}{\frac{dz}{d\eta}} = -\left\{\frac{d}{d\eta}\left[(1-f)\sqrt{R_e^2-\eta^2}\right]\right\}^{-1} \quad (2\text{-}6)$$

$$\tan\phi = \frac{1}{(1-f)^2}\frac{z}{\eta}\ ; \ -\pi/2 \le \phi \le \pi/2 \quad (2\text{-}7)$$

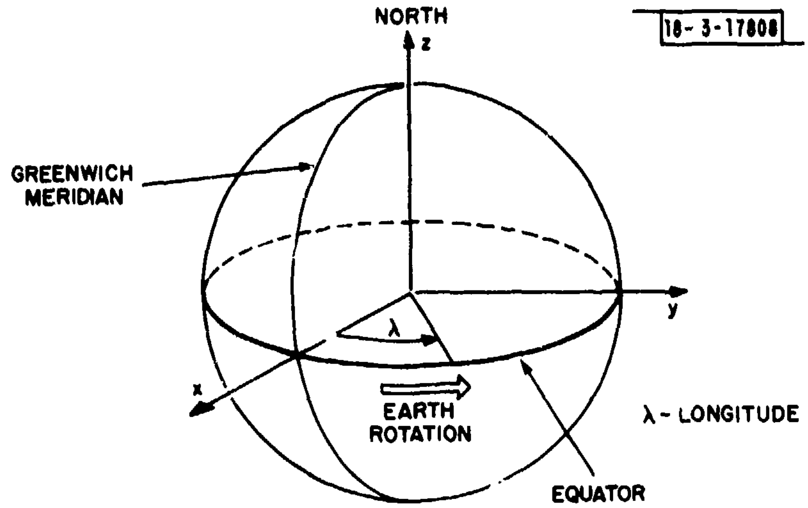

Fig. 2.1. Oblique view of earth.

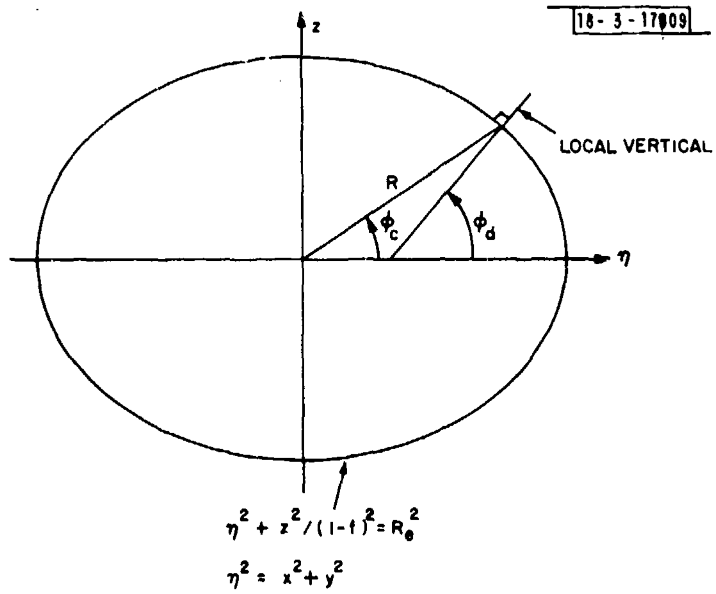

Fig. 2.2. Cross-sectional view of earth.

Note that the extension of the local normal to the equatorial plane does not in general pass through the center of the earth (exceptions occur when $\phi=0$ and $\phi=\pi/2$). The geodetic latitude is the usual survey latitude. It is also the latitude determined from astronomic observations.

Geocentric latitude ϕ_c, on the other hand, is simply the angle formed by the equatorial plane and a vector from the center of the earth to a point on the earth surface. It is defined by

$$\tan\phi_c = \frac{z}{\eta} \tag{2-8}$$

or, in terms of geodetic latitude,

$$\tan\phi_c = (1-f)^2 \tan\phi \tag{2-9}$$

The radius of the earth at an arbitrary location on the surface can be expressed as

$$R = \sqrt{\eta^2+z^2} = \frac{R_n}{\sqrt{1-\varepsilon^2\cos^2\phi_c}} \tag{2-10}$$

or, in terms of geodetic latitude,

$$R = R_e\sqrt{1-\varepsilon^2\sin^2\phi} \tag{2-11}$$

It is frequently necessary to compute the altitude of a target above the earth ellipsoid. That is, for a target at a location (η_o,z_o) outside the

ellipse, it is desired to find the minimum of

$$h = \sqrt{(\eta_o-\eta)^2 + (z_o-z)^2} \tag{2-12}$$

subject to the constraint that (η,z) is on the surface of the ellipsoid:

$$\eta^2 + \frac{z^2}{(1-f)^2} = R_e^2 \tag{2-13}$$

A direct attack on this minimization leads to a quartic equation which is very difficult to solve. Fortunately, the eccentricity of the earth ellipsoid is quite small and a reasonably accurate approximation can be made. This approximation is simply to take the difference between the distance from the center of the earth to the target and the radius of the earth in the same direction:

$$h_o = r_o - \frac{R_n}{\sqrt{1-\varepsilon^2\cos^2A}} \tag{2-14}$$

where

$$r_o = \sqrt{\eta_o^2 + z_o^2} \tag{2-15}$$

and

$$\cos A = \frac{\eta_o}{r_o} \tag{2-16}$$

A detailed evaluation of the error in this simple relation for h_o has not been made; however, sample computations indicated worst-case errors of the order of

10 meters for altitudes around 2000 Km and about 4 meters for altitudes around 700 Km.

For the purposes of the ARIES Program, the above model for h_o is probably adequate. However, a refinement of the relation can be made by taking first order perturbations of η and z around h_o. Such a procedure yields

$$h_1 = h_o \frac{(1-\varepsilon^2\cos^2 A)}{\sqrt{1-(2-\varepsilon^2)\varepsilon^2\cos^2 A}} \tag{2-17}$$

Sample calculations indicate worst-case errors of only 3 meters for h = 2000 Km and only 0.5 meters for h = 700 Km.

Assorted coordinate systems associated with the earth model and appropriate transformations between each pair of these coordinate systems will be defined in Sections 3 and 5.

2.2 Earth Gravitational Field

The gravitational potential of the earth is generally expanded in a series of spherical harmonics. The most important component of the potential is the point mass potential

$$V_o(r) = \frac{G}{r} \tag{2-18}$$

where the gravitational constant G is

$$G = 3.986012 \times 10^{14} \text{ (meters)}^3/\text{(second)}^2 \tag{2-19}$$

and r is the radius from the center of the earth. Additional contributions to the gravitational potential result from the mass distribution of the earth. These perturbations of the point mass potential are expanded in spherical harmonics. One series of harmonics depends only on latitude; these are called the zonal harmonics. All remaining terms of the harmonic series have a dependence on both latitude and longitude; these are called tesseral harmonics.

The ARIES Program only considers the zonal harmonics in the gravitational potential model. That is, the potential is given by

$$V(r) = \frac{G}{r}\left[1 - \sum_{n=2}^{9} \left(\frac{R_e}{r}\right)^n C_n P_n(\sin A)\right] \tag{2-20}$$

where C_n is the coefficient of the $n^{\underline{th}}$ zonal harmonic and $P_n(\cdot)$ is the Legendre polynomial of the first kind of order n. The angle A is the angle between the equatorial plane and the vector from the center of the earth to the target position. The first order harmonic is zero as a result of the symmetry of the gravitational field. Since the coefficient of the second zonal harmonic is roughly three orders of magnitude greater than all other coefficients, the ARIES Program is further simplified to include only the point mass potential and the second zonal gravitational harmonic. (Additional zonal harmonic coefficients, through the ninth, are available in the ARIES Program; they are easily implemented in Subroutine GRAVTY.)

Values for the harmonic coefficients in the gravitational potential model are

$$
\begin{aligned}
C_2 &= 1.08271 \times 10^{-3} \\
C_3 &= -2.630 \times 10^{-6} \\
C_4 &= -2.35 \times 10^{-6} \\
C_5 &= -0.265 \times 10^{-6} \\
C_6 &= 0.66 \times 10^{-6} \\
C_7 &= -0.46 \times 10^{-6} \\
C_8 &= +0.53 \times 10^{-6} \\
C_9 &= 0.244 \times 10^{-6}
\end{aligned}
\qquad (2\text{-}21)
$$

SECTION 3--COORDINATE SYSTEMS

Several different coordinate systems are used in the ARIES Program. The various operations performed in the Program (e.g., trajectory integration, target tumbling, radar measurement generation) all have preferred coordinate frames for their evaluation. In the sections below the various coordinate systems employed by the ARIES Program are defined. The transformations between the various coordinate systems are presented in Section 5.

3.1 Earth Centered Inertial Frame (ECI)

A primary inertial frame employed by the ARIES Program is the Earth Centered Inertial (ECI) frame. The origin of this frame is at the center of the earth, the z-axis passes through the North Pole, and the equatorial plane of the earth lies in the x-y plane. At the reference time (typically the launch time of the target), the x-axis of the ECI frame passes through the Greenwich meridian. The x, y, and z axes shown in Fig. 2.1 form an ECI frame if the axes are held fixed in space rather than rotating with the earth.

The equations of motion for ballistic targets have their simplest form in the ECI frame. Consequently, it is the preferred frame for trajectory generation and trajectory integration.

3.2 Earth Centered Fixed Frame (ECF)

This frame is essentially the same as the ECI frame, except that it rotates with the earth. Usually the x-axis is defined to point through the

equator at the Greenwich meridian and the y-axis is defined to point through the equator at 90° East longitude.

The ECF frame is a convenient frame for the definition of locations on the earth surface. Radar sites, target launch points and target impact points are fixed points in an ECF frame.

3.3 Radar Coordinate Systems

There are four different coordinate systems associated with the radars. Two of these are peculiar to a phased array radar (X_{RF}, Y_{RF}, Z_{RF} and R, U, V). One coordinate system (R, A, E) is associated with a conventional, mechanically steered radar. The fourth coordinate system is an earth surface fixed frame or radar XYZ frame. Radar measurements are typically made in either RAE or RUV coordinates. The cartesian frames are useful for many computations and provide the necessary cartesian frames for definition of RAE and RUV coordinates.

3.3.1 Radar Cartesian Coordinates (XYZ)

The Radar Cartesian Frame is defined such that the X-Y plane is tangent to the earth ellipsoid and the Z-axis is pointed outward along the local vertical. The X-axis points East and the Y-axis points North. This set of coordinates is illustrated in Fig. 3.1. Usually the origin of this coordinate frame will coincide with the radar location or with the target launch point. If the radar is not located on the earth ellipsoid, but is actually at height H above the ellipsoid, then the radar X-Y plane will be parallel to the tangent plane to the earth ellipsoid.

3.3.2 Radar RAE Coordinates

The RAE coordinates are defined relative to the radar cartesian coordinates as indicated in Fig. 3.2. R is the distance (range) from the origin (the radar) to the target. The azimuth, A, is the angle from the Y-axis (North) to the projection of the range vector, R, into the X-Y plane. Azimuth is measured clockwise from North toward East. The elevation angle, E, is measured (positive up) from the X-Y plane to the range vector, R.

3.3.3 Phased Array Radar Face Cartesian Coordinates

The cartesian coordinates (X_{RF}, Y_{RF}, Z_{RF}) are defined with respect to the phased array radar face as indicated in Fig. 3.3. The radar face lies in the X_{RF}-Y_{RF} plane, with the X_{RF}-axis horizontal and the Y_{RF}-axis pointing upward. The Z_{RF}-axis points outward along the normal to the array face.

The orientation of the phased array face is defined by the azimuth and the elevation of the phased array boresight (i.e., the phased array Z-axis). Azimuth and elevation, in this case, are measured in the same sense as defined above for radar RAE coordinates.

3.3.4 Phased Array RUV and Rαβ Coordinates

The RUV and Rαβ coordinate systems are defined relative to the Phased Array Radar Face Cartesian Frame. R is the distance (range) from the origin (the radar) to the target. Target direction in angle is measured by the direction cosines with respect to the X_{RF} and Y_{RF} axes. These direction

cosines are designated U and V, respectively. Note that conical surfaces about the X_{RF} and about the Y_{RF} axes are defined by U = constant and by V = constant; the intersection of these cones defines the direction to the target. (In the usual spherical coordinate system ($R\Theta\phi$) Θ defines a cone about the z-axis and ϕ defines a plane containing the z-axis; the intersection of the conical surface and the plane defines target direction.)

The angles α and β can also be used to measure the direction to a target with respect to the Phased Array Radar Face Cartesian Frame. In this case, α and β are the arcsines of U and V, respectively. Note that $(\pi/2-\alpha)$ and $(\pi/2-\beta)$ are the direction cosine <u>angles</u> with respect to the X_{RF} and Y_{RF} axes. The phased array broadside direction is characterized by U = V = 0, which corresponds to direction cosine angles of $\pi/2$ (and, therefore, $\alpha = \beta = 0$).

3.4 Target Coordinate Systems

The target tumbling and RCS generation programs utilize several different coordinate systems which are unique to this particular subject. In particular, the following frames are employed:

1. Principal body axes frame - this frame has its origin at the center of mass of the object and rotates with the object.
2. Momentum frame - this is an inertial frame with its z-axis aligned with the momentum vector. (The momentum vector is invariant in an inertial frame.)

3. Local horizontal frame - this frame has its z-axis in the direction of the ECI velocity vector and its y-axis normal to the trajectory plane.

These frames are inter-related by the target Euler angles. The local horizontal frame is directly defined from the target position and the target velocity in the ECI frame.

Complete details about these coordinate systems and transformations among them are presented in Section 7 of Reference 2.

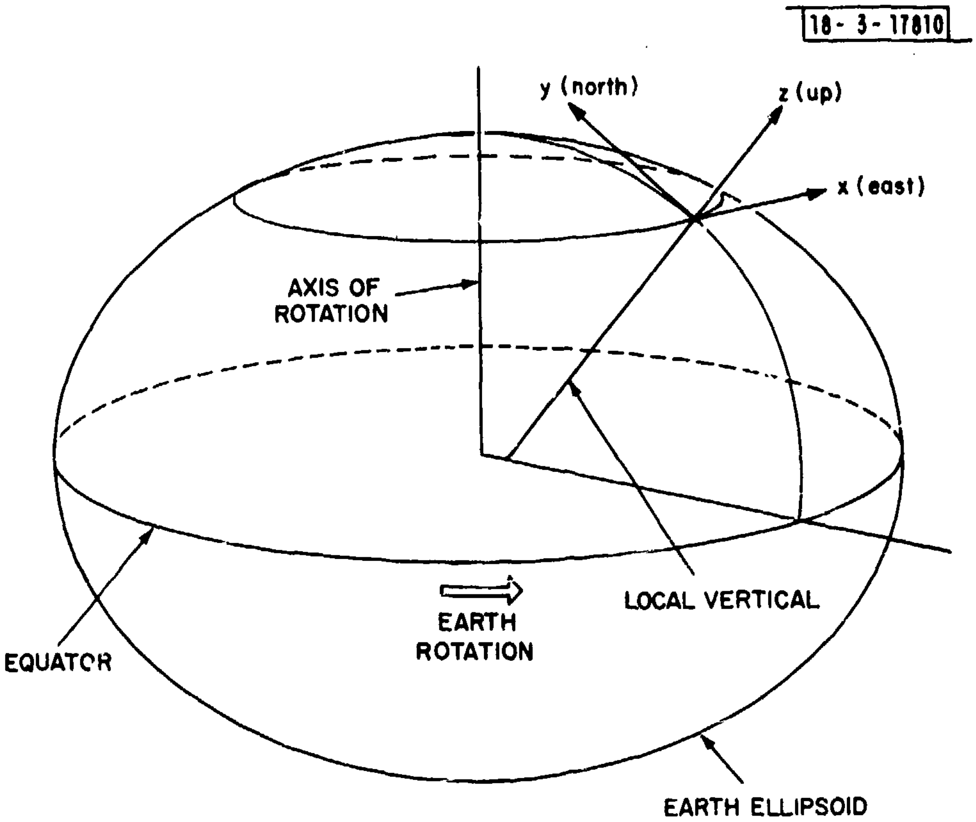

Fig. 3.1. Radar cartesian coordinates (XYZ).

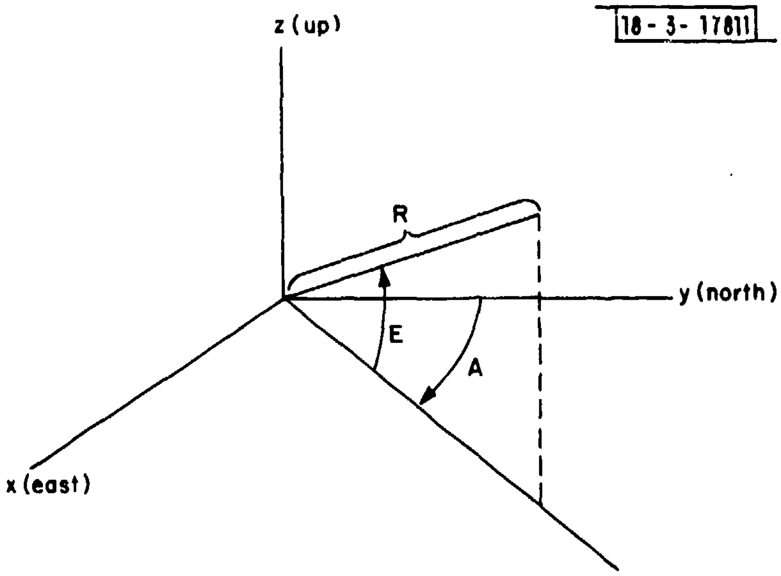

Fig. 3.2. Radar RAE coordinates.

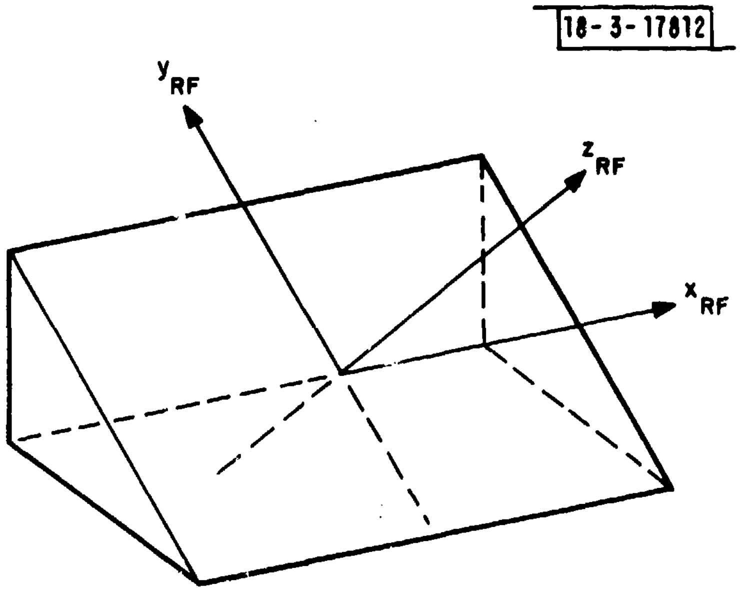

Fig. 3.3. Phased array radar face cartesian coordinates.

SECTION 4--BALLISTIC EQUATIONS OF MOTION

In this Section the equations of motion for a target on a ballistic trajectory are presented. The first part gives the required relationship for target motion in the presence of only the gravitational field of the earth.

Atmospheric drag effects are not included in the acceleration model for the ARIES Program, but it seems desirable to explain how they might be added to the model. This is done in Section 4.2.

The last part (4.3) explains the predictor-corrector technique for trajectory integration in the ARIES Program.

4.1 Target Acceleration Due to Gravity

A target located at a position (x,y,z) outside the surface of the earth experiences gravitational forces which result in the following target accelerations*

$$\ddot{x} = \nabla V(r) \tag{4-1}$$

where ∇ is the gradient operator

$$\nabla = i_x \frac{\partial}{\partial x} + i_y \frac{\partial}{\partial y} + i_z \frac{\partial}{\partial z} \tag{4-2}$$

and V(r) is the gravitational potential function defined in Section 2.2. Target position is given in the ECI coordinate frame. The three cartesian components of the acceleration vector can be shown to be:

*Lower case script characters denote vector (or column matrix) quantities. Upper case script characters denote matrices.

$$\ddot{x} = -\frac{Gx}{r^3} g_1 \tag{4-3}$$

$$\ddot{y} = -\frac{Gy}{r^3} g_1 \tag{4-4}$$

$$\ddot{z} = -\frac{Gz}{r^3} g_1 - \frac{G}{r^3} g_2 \tag{4-5}$$

where

$$g_1 = 1 - \sum_{n=2}^{9} \left(\frac{R_e}{r}\right)^n C_n P'_{n+1}(\sin A) \tag{4-6}$$

$$g_2 = \sum_{n=2}^{9} \left(\frac{R_e}{r}\right)^n C_n P'_n(\sin A) \tag{4-7}$$

In these relations $P'_n(\cdot)$ is the derivative of the Legendre polynomial with respect to its argument. Also,

$$r = \sqrt{x^2+y^2+z^2} \tag{4-8}$$

and

$$\sin A = \frac{z}{r} \tag{4-9}$$

The following recursion relation for the derivatives of the Legendre polynomials

$$P'_{n+1}(\alpha) = (2 + \frac{1}{n})\left[\alpha P'_n(\alpha) - P'_{n-1}(\alpha)\right] + P'_{n-1}(\alpha) \tag{4-10}$$

is useful for the computation of g_1 and g_2. Note also that

$$P_0'(\alpha) = 0, \; P_1'(\alpha) = 1 \text{ and } P_2'(\alpha) = 3\alpha. \tag{4-11}$$

Given the target position and the target velocity in the ECI frame at a particular reference time, together with the above definitions for target accelerations, the motion of the target in the ECI frame is completely defined over the entire trajectory. This statement, of course, ignores any effects due to atmospheric drag.

Subroutine GRAVTY is used to compute the gravitational acceleration components of a target state vector.

4.2 Target Acceleration Due to Atmospheric Drag

The ARIES Program does not include the target deceleration effects caused by atmospheric drag. However, for completeness, a brief discussion of atmospheric drag is included.

Atmospheric drag results in a component of acceleration in the direction of the velocity vector given by:

$$a_D = - \frac{1}{2} C_D \, A \, \rho \, v^2/(w/g) \tag{4-12}$$

where

C_D = coefficient of drag of the reentry body

A = cross-sectional area of the reentry body

ρ = atmospheric density

v = body velocity (magnitude)

w = body weight

g = acceleration of gravity

The parameters C_D, A and w depend on the shape and size of the target and would have to be inputs to the ARIES Program. Atmospheric density is a function of several parameters including altitude, temperature, latitude, and longitude. An appropriate atmospheric density profile would also have to be input to the program, either as an analytic formula or as a table. Body velocity is in either an ECF frame or an earth surface fixed (i.e., radar XYZ) frame. With the following definition

$$C_D' = C_D/v \tag{4-13}$$

target acceleration due to drag can be written as

$$\begin{aligned} a_{Dx} &= -C_D' \, v_x \\ a_{Dy} &= -C_D' \, v_y \\ a_{Dz} &= -C_D' \, v_z \end{aligned} \tag{4-14}$$

where v_x, v_y and v_z are the three cartesian components of velocity in an ECF frame. These accelerations must then be transformed to the ECI frame and added to those due to gravity to obtain total target acceleration in the atmosphere. (The atmosphere is typically defined to cover the altitude regime from 0 to 300 Kft (91.44 Km). Re-entry altitude is defined to be 300 Kft in the ARIES Program.)

The addition of atmospheric drag would result in a better modeling of target motion. However, it would add considerable complexity to the ARIES Program, particularly in the trajectory generation and trajectory estimation areas, without any significant impact on the radar measurement modeling which is a major part of the ARIES Program.

4.3 Predictor-Corrector Trajectory Integration

Trajectory integration in the ARIES Program is performed with a predictor-corrector integration scheme. The procedure for integration from $t_1 = 0$ to $t_2 = \Delta t$ is explained below.

If the accelerations at the beginning, $a(0)$, and end, $a(\Delta t)$, of the interval were known, the acceleration over the interval could be approximated by a straight line:

$$a(t) = a(0) + \frac{a(\Delta t)-a(0)}{\Delta t}\, t \tag{4-15}$$

The position and velocity at time Δt are then obtained from

$$x(\Delta t) = x(0) + v(0)\Delta t + \int_0^{\Delta t} dt \int_0^{t} a(\tau)d\tau \tag{4-16}$$

$$\begin{aligned} &= x(0) + v(0)\Delta t + a(0)(\Delta t)^2/2 \\ &\quad + [a(\Delta t)-a(0)](\Delta t)^2/6 \end{aligned} \tag{4-17}$$

and

$$v(\Delta t) = v(0) + a(0)\Delta t + [a(\Delta t)-a(0)]\Delta t/2 \tag{4-18}$$

In practice, of course, $a(\Delta t)$ is not known so the integration must be done in two steps. First, based on the known target position at $t = t_1$, the accelerations a_x, a_y and a_z are computed from the gravity model. Then, target position and velocity are <u>predicted</u> for time $t = t_2 = t_1 + \Delta t$:

$$x_p(t_1+\Delta t) = x(t_1) + v(t_1)\Delta t + a(t_1)(\Delta t)^2/2 \tag{4-19}$$

$$v_p(t_1+\Delta t) = v(t_1) + a(t_1)\Delta t \tag{4-20}$$

From the predicted position at $t = t_2$, the end-point accelerations are computed. Then a final <u>correction</u> of the predicted (integrated) position and velocity is made:

$$x(t_2) = x_p(t_2) + [a(t_2)-a(t_1)](\Delta t)^2/6 \tag{4-21}$$

$$v(t_2) = v_p(t_2) + [a(t_2)-a(t_1)]\Delta t/2 \tag{4-22}$$

This procedure is, of course, performed simultaneously on all three cartesian coordinates of the ECI state vector.

The extrapolation (integration) of a target state vector from t_1 to t_2 is performed by Subroutine EXTRAP (X,T2). The components of the input target state vector X are:

X(1) = T1, valid time of the state vector

X(2-4) = ECI positions in x, y, z

X(5-7) = ECI velocities in x, y, z

X(8-10) = ECI accelerations in x, y, z

X(11) = target altitude

Integration is performed with a maximum step size of 1 second to prevent a buildup of errors. (Tests have indicated that $\Delta t = 1$ second gives very accurate results.) The target state vector is valid for the time T2 when it is returned to the calling program from Subroutine EXTRAP.

SECTION 5--COORDINATE TRANSFORMATIONS

The ARIES Program utilizes several different coordinate systems as defined in Section 3. Results of operations performed in one coordinate system are frequently required in a different coordinate system. For example, a trajectory is integrated in an ECI frame, but it is used in a Radar RAF (or RUV) frame for the generation of radar measurements.

This Section presents all coordinate transformations relevant to trajectory generation, trajectory estimation and tracking error evaluation. Some additional transformations involving body coordinate systems for RCS generation are given in Ref. 2, Section 7.

5.1 Between ECI and ECF

At the reference time (usually t = 0), the ECI and ECF frames are coincident. Both have their x-axes through the equator at the Greenwich meridian and their z-axes through the North Pole. At a later time t the ECI frame is unchanged but the ECF frame has rotated with the earth through an angle $\omega_e t$. Let the subscript I indicate a component in the ECI frame and the subscript F a component in the ECF frame. Now it is easily shown that the rotation around the z-axis by an amount $\omega_e t$ results in the transformation

$$\begin{bmatrix} x_F \\ y_F \\ z_F \end{bmatrix} = \begin{bmatrix} \cos\omega_e t & \sin\omega_e t & 0 \\ -\sin\omega_e t & \cos\omega_e t & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_I \\ y_I \\ z_I \end{bmatrix} \tag{5-1}$$

or, by defining the coefficient matrix as $\mathcal{G}(t)$ and the position vectors as x_I and x_F:*

$$x_F = \mathcal{G}(t) x_I \tag{5-2}$$

If one takes the time derivative of Eqn. (5-1), the following transformation of the velocity vector is obtained:

$$\dot{x}_F = \mathcal{G}(t)\left\{\dot{x}_I + \begin{bmatrix} 0 & \omega_e & 0 \\ -\omega_e & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} x_I\right\} = \mathcal{G}(t)\begin{bmatrix} \dot{x}_I + \omega_e y_I \\ \dot{y}_I - \omega_e x_I \\ \dot{z}_I \end{bmatrix} \tag{5-3}$$

where the dot over a vector (or, a vector component) indicates a derivative with respect to time. The second term in Eqn. (5-3) results from the continuous rotation of the ECF frame relative to the inertial frame. For the ARIES Program we require only position and velocity transformations between ECI and ECF frames. Target accelerations are used only for trajectory integration (ECI frame) and therefore do not have to be transformed.

The transformation matrix $\mathcal{G}(t)$ represents an orthonormal transformation. Consequently, $\mathcal{G}(t)$ is orthogonal; i.e., the inverse is equal to the transpose:

$$\mathcal{G}(t)^{-1} = \mathcal{G}(t)^T$$

The reverse transformation, from ECF to ECI, is therefore given by

$$x_I = \mathcal{G}(t)^T x_F \tag{5-4}$$

*Script characters denote matrices (lower case for column matrices; upper case for N x M matrices).

for positions and by

$$\dot{x}_I = G(t)^T \left\{ \dot{x}_F - \begin{bmatrix} 0 & \omega_e & 0 \\ -\omega_e & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} x_F \right\} = G(t)^T \begin{bmatrix} \dot{x}_F - \omega_e y_F \\ \dot{y}_F + \omega_e x_F \\ \dot{z}_F \end{bmatrix} \tag{5-5}$$

for velocities.

Subroutine ECIECF (XI,XF,TIME) is used to transform the state vector XI in ECI coordinates to the state vector XF in ECF coordinates. TIME is the interval in seconds since the two frames were coincident. Subroutine ECFECI(XF, XI,TIME) performs the inverse transformation.

5.2 Between ECF and Radar XYZ

Consider a radar located at geodetic latitude ϕ, longitude λ (measured east from the Greenwich meridian) and at height h above the earth ellipsoid. The process of transforming from the ECF frame to a radar XYZ (East, North, Up) frame can be visualized as the following sequence of operations:

1. Rotation by angle λ about the z-axis.
2. Rotation by angle ϕ_c about the y-axis.
3. Translation by the local earth radius along the x-axis.
4. Rotation by angle $(\phi - \phi_c)$ about the y-axis.
5. Translation by h along the x-axis.
6. Coordinate interchange to obtain the desired East, North, Up sequence.

The appropriate matrices for these operations are:

1.
$$\mathcal{B} = \begin{bmatrix} \cos\lambda & \sin\lambda & 0 \\ -\sin\lambda & \cos\lambda & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{5-6}$$

2.
$$\mathcal{C} = \begin{bmatrix} \cos\phi_c & 0 & \sin\phi_c \\ 0 & 1 & 0 \\ -\sin\phi_c & 0 & \cos\phi_c \end{bmatrix}, \text{ where } \tan\phi_c = (1-\varepsilon^2)\tan\phi \tag{5-7}$$

3.
$$r_e = \begin{bmatrix} -R_{\epsilon h} \\ 0 \\ 0 \end{bmatrix}, \text{ where } R_{eh} = \frac{R_n}{\sqrt{1-\varepsilon^2\cos^2\phi_c}} \tag{5-8}$$

4.
$$\mathcal{D} = \begin{bmatrix} \cos(\phi-\phi_c) & 0 & \sin(\phi-\phi_c) \\ 0 & 1 & 0 \\ -\sin(\phi-\phi_c) & 0 & \cos(\phi-\phi_c) \end{bmatrix} \tag{5-9}$$

5.
$$h = \begin{bmatrix} -h \\ 0 \\ 0 \end{bmatrix} \tag{5-10}$$

6.
$$\mathcal{E} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \tag{5-11}$$

If the subscript R refers to the radar frame, then we can write

$$x_R = \{EDCB\}x_F + E\{h+Dr_e\} \tag{5-12}$$

When the indicated matrix multiplications are performed, the resulting relations are:

$$x_R = Ax_F + b \tag{5-13}$$

where

$$A = \begin{bmatrix} -\sin\lambda & \cos\lambda & 0 \\ -\sin\phi\ \cos\lambda & -\sin\phi\ \sin\lambda & \cos\phi \\ \cos\phi\ \cos\lambda & \cos\phi\ \sin\lambda & \sin\phi \end{bmatrix} \tag{5-14}$$

and

$$b = \begin{bmatrix} 0 \\ R_{eh}\ \sin(\phi-\phi_c) \\ -R_{eh}\ \cos(\phi-\phi_c)-h \end{bmatrix} \tag{5-15}$$

Since b simply represents a translation, it does not enter into the velocity transformation. For the velocity transformation from ECF to radar XYZ, we have simply:

$$\dot{x}_R = A\dot{x}_F \tag{5-16}$$

The matrix A is orthogonal, so the inverse transformation, Radar XYZ to ECF, is

$$x_F = A^T x_R - A^T b \tag{5-17}$$

and

$$\dot{x}_F = A^T \dot{x}_R \tag{5-18}$$

The transformation matrices A and b are initially computed in Subroutine SETUP(W,A) where W is a three vector containing radar longitude, geodetic latitude and height above the earth ellipsoid. The A and b matrices are returned in the transfer vector A. This vector A is then passed to Subroutines ECFXYZ(XF,XR,A) and XYZECF(XR,XF,A) to perform the actual coordinate transformations.

5.3 Between Radar XYZ and RAE

The Radar Range, Azimuth and Elevation are defined in Section 3.3.2. They are related to the Radar XYZ coordinates as follows:

$$\begin{aligned} R &= \sqrt{x^2+y^2+z^2} \\ A &= \tan^{-1}(x/y), \quad 0 \le A \le 2\pi \\ E &= \sin^{-1}(z/R), \quad -\pi/2 \le E \le \pi/2 \end{aligned} \tag{5-19}$$

The proper quadrant must be determined for A so that A ranges from 0 to 2π. Time derivatives of R, A and E yield:

$$\dot{R} = \frac{(x\dot{x}+y\dot{y}+z\dot{z})}{R}$$

$$\dot{A} = \frac{(y\dot{x}-x\dot{y})}{x^2+y^2}$$

$$\dot{E} = \frac{(\dot{z}R-z\dot{R})}{R\sqrt{x^2+y^2}} \qquad (5\text{-}20)$$

These transformations from Radar XYZ to Radar RAE are performed by Subroutine XYZRAE.

The transformation from RAE to XYZ is performed by Subroutine RAEXYZ with the use of the following relations:

$$x = R \sin A \cos E$$

$$y = R \cos A \cos E$$

$$z = R \sin E \qquad (5\text{-}21)$$

and

$$\dot{x} = S \sin A + y\dot{A}$$

$$\dot{y} = S \cos A - x\dot{A}$$

$$\dot{z} = \dot{R} \sin E + R\dot{E} \cos E \qquad (5\text{-}22)$$

where

$$S = \dot{R} \cos E - R\dot{E} \sin E \qquad (5\text{-}23)$$

5.4 Between Radar XYZ and Phased Array XYZ

The definition for the Phased Array XYZ coordinate system was given previously in Section 3.3.3. Basically, the orientation of the array face is defined by the azimuth A_p and the elevation E_p of the normal to the array face. The array face lies in the x_p-y_p plane with x_p horizontal and y_p up. The z_p axis is aligned with the array normal. (The subscript p has the same meaning as the subscript RF used in Sections 3.3.3 and 3.3.4.)

If we start with the array face coordinates coincident with the radar XYZ frame (that is, the array is pointed up), then the following sequence of transformations gives the desired Radar XYZ to Phased Array XYZ transformation:

$$\begin{bmatrix} x_p \\ y_p \\ z_p \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sin E_p & -\cos E_p \\ 0 & \cos E_p & \sin E_p \end{bmatrix} \begin{bmatrix} \cos A_p & -\sin A_p & 0 \\ \sin A_p & \cos A_p & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_R \\ y_R \\ z_R \end{bmatrix} \tag{5-24}$$

or,

$$x_p = A_{rp} x_R \tag{5-25}$$

where

$$A_{rp} = \begin{bmatrix} -\cos A_p & \sin A_p & 0 \\ -\sin A_p \sin E_p & -\cos A_p \sin E_p & \cos E_p \\ \sin A_p \cos E_p & \cos A_p \cos E_p & \sin E_p \end{bmatrix} \tag{5-26}$$

Note that when $A_p = E_p = 0$ (i.e., the phased array is pointed north), $x_p = -x_R$, $y_p = z_R$ and $z_p = y_R$, as would be expected from the definition of the phased array XYZ coordinate system.

The velocity transformation is simply

$$\dot{x}_p = A_{rp}\dot{x}_R \tag{5-27}$$

Since the matrix A_{rp} is orthogonal, the transformation matrix for converting from Phased Array XYZ to Radar XYZ is simply the transposed matrix A_{rp}^T.

Subroutine XYZXRF(XR,XP,Y) is used to convert a state vector x_R in Radar XYZ to a state vector x_p in Phased Array XYZ coordinates. The reverse transformation is performed by Subroutine XRFXYZ(XP,XR,Y). In both Subroutines the two element vector Y contains the azimuth A_p and elevation E_p of the phased array normal.

5.5 Between Phased Array XYZ and RUV

The Phased Array range and "angle" coordinates RUV are defined by:

$$R = \sqrt{x_p^2+y_p^2+z_p^2}$$

$$U = x_p/R = \sin\alpha, \ -1 \le U \le 1$$

$$V = y_p/R = \sin\beta, \ -1 \le V \le 1 \tag{5-28}$$

The velocity relations are obtained by taking derivatives of Eqn. (5-28):

$$\dot{R} = (x_p\dot{x}_p+y_p\dot{y}_p+z_p\dot{z}_p)/R$$

$$\dot{U} = (\dot{x}_p-U\dot{R})/R$$

$$\dot{V} = (\dot{y}_p-V\dot{R})/R \tag{5-29}$$

The transformation from Phased Array XYZ coordinates to Phased Array RUV coordinates is performed by Subroutine XRFRUV(XP,RUV).

The inverse transformation, from RUV to Phased Array XYZ, has the following relations:

$$X = RU$$

$$Y = RV$$

$$Z = R\sqrt{1-U^2-V^2} = RW \tag{5-30}$$

and

$$\dot{X} = \dot{R}U + R\dot{U}$$

$$\dot{Y} = \dot{R}V + R\dot{V}$$

$$\dot{Z} = \dot{R}W - \frac{R(U\dot{U}+V\dot{V})}{W} \tag{5-31}$$

This transformation is performed by Subroutine RUVXRF(RUV,XP).

5.6 Target Coordinate Systems

The target tumbling and RCS generation programs utilize several different coordinate systems which are unique to this particular subject. These coordinate systems are briefly defined in Section 3.4. A complete description of

these coordinate systems and the transformations among them are presented in Section 7 of Reference 2.

5.7 Other Combinations

The coordinate transformations defined above consider only transformations between quite closely related coordinate systems. Other transformations may be obtained by the use of a sequence of the above transformations. For example, to transform a target state vector in the ECI frame to a target state vector in the Phased Array RUV frame, the following sequence of transformations would be performed:

1. ECI to ECF (ECIECF)
2. ECF to Radar XYZ (ECFXYZ)
3. Radar XYZ to Phased Array XYZ (XYZXRF)
4. Phased Array XYZ to RUV (XRFRUV)

The names in parentheses are the names of the ARIES Subroutines which perform the indicated transformations. This particular transformation (ECI to Phased Array RUV) and the transformation from ECI to Radar RAE, are quite important and, consequently, they are available by calling a single Subroutine ECIRAD (XI,W,N,TYPE). XI is the input state vector in ECI coordinates, W is the output state vector in Radar coordinates, N is the identification number of the Radar (for use in multiple Radar coverage) and TYPE identifies the radar as either a dish radar or a phased array radar.

The coordinate transformations defined in Sections 5.1 to 5.5 provide all the information required to transform a state vector from one frame to any other frame. More than one subroutine may have to be called, as in the above example, but the key point is that the capability does exist.

5.8 Radar Coverage Limits

A conventional dish radar usually has the capability of illuminating and tracking a target over the entire hemisphere above the local horizon plane. On the other hand, a phased array is usually a permanently fixed installation (the array normal direction does not change). Consequently, the coverage of a phased array radar face is limited to a fraction of the hemisphere. Typical phased array coverage lies in the range of 1/4 to 1/3 of a hemisphere.

For a phased array radar, in particular, it is important to test the target state vector to determine whether the target lies within the assumed field of view of the phased array. This test is performed on the RAE target state vector in Subroutine LIMITS(N,RAE,FLAG). The radar number N and the target state vector in RAE coordinates are inputs to the subroutine. FLAG=1 is returned if the target is in the field of view; FLAG=0, otherwise. For a dish radar, the test simply amounts to checking that target elevation is greater then zero. A phased array radar, on the other hand, is assumed to have the capability of viewing a sector in azimuth and elevation about the boresight pointing direction. For example, if the boresight azimuth of the phased array is A_p, then one-third hemisphere coverage is achieved by applying azimuth bounds of $A_p - \pi/3$ and $A_p + \pi/3$ and elevation bounds of 0 and $\pi/2$. Both

azimuth and elevation are checked against the phased array radar coverage bounds to determine whether the target is within the field of view. Note that it is easier to define phased array coverage limits in azimuth and elevation, rather than the normal phased array U and V coordinates.

SECTION 6--TRAJECTORY GENERATION

The ARIES Program requires the capability of generating a wide range of target trajectories to simulate various radar/target geometries. For the present requirements, it is sufficient to consider only target trajectories launched from and impacting on the surface of the earth. In addition, atmospheric drag is ignored and the trajectories take the short way around the earth from launch to impact.

The trajectory generation process requires the following inputs:

1. Launch longitude and geodetic latitude
2. Impact longitude and geodetic latitude
3. Type of trajectory
   a. Launch angle specified, or
   b. Re-entry angle specified, or
   c. Minimum energy

The procedure used in the generation of a trajectory is to first get a good trajectory estimate using Kepler's equations for satellite motion in a central force field. Perturbations of this first estimate are then made to account for the zonal harmonics of the gravitational field and to account for the oblateness of the earth ellipsoid.

Section 6.1 contains a brief summary of the equations which define the motion of an object in an elliptic orbit around the earth (central force field only). This is followed by a discussion of initialization of the trajectory

state vector subject to the launch, impact and type of trajectory constraints in Section 6.2. Finally, the procedures employed in perturbing the initial state vector to obtain the desired precise state vector are presented in Section 6.3.

6.1 Equations for an Elliptic Orbit

In this Section, a set of equations is developed for target motion in an elliptic orbit which intercepts the surface of the earth. Figure 6.1 illustrates the geometry of the problem as viewed in the plane of the trajectory. (Note that this plane is only defined in the ECI frame.) Since two of the trajectory types considered involve specification of angles (launch angle or re-entry angle), the following derivations of the orbit parameters contain launch elevation as an explicit variable.

Most standard physics textbooks contain the derivation of the orbital motion of an object in the presence of the point mass gravitational potential. They are summarized here for the gravitational potential of the earth.

Radius (center of earth to ellipse):

$$r = \frac{a(1-e^2)}{1-e\cos\Theta} \tag{6-1}$$

where a is the semi-major axis, e is the eccentricity of the orbit and Θ is the central angle measured from apogee (see Fig. 6.1). Note that the true anomaly is measured from perigee and is equal to $(\pi-\Theta)$. The radius can also be defined in terms of the eccentric anomaly E_c:

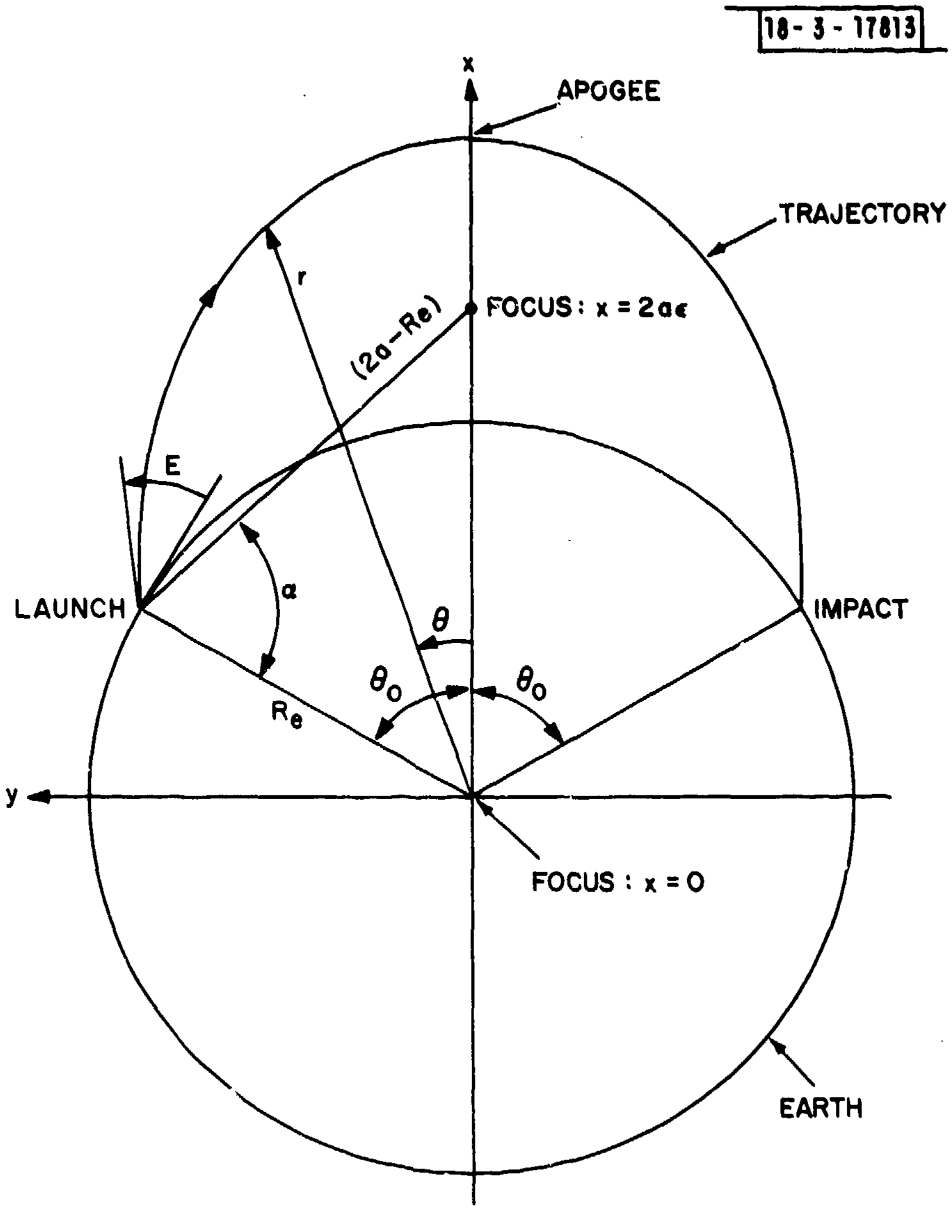

Fig. 6.1. Geometry of an elliptic trajectory as viewed in the orbit plane.

$$r = a(1-e\cos E_c) \tag{6-2}$$

If these two definitions for r are equated, it can be shown that true anomaly and eccentric anomaly are related by:

$$\tan(E_c/2) = \sqrt{\frac{1-e}{1+e}}\tan\left(\frac{\pi-\Theta}{2}\right) \tag{6-3}$$

Two other parameters are also of interest:

Mean motion:

$$n = \sqrt{G/a^3} \tag{6-4}$$

Mean anomaly:

$$M = n(t-\tau) \tag{6-5}$$

or,

$$M = E_c - e\sin E_c \tag{6-6}$$

where τ is the time of passing perigee and G is the gravitational constant defined by Eqn. (2-19). For a full orbit, E_c ranges from 0 to 2π. M also goes from 0 to 2π, so the orbital period is

$$T = 2\pi/n \tag{6-7}$$

The total energy and the angular momentum are constants for the orbit. That is,

$$\tfrac{1}{2}m(\dot{r}^2+r^2\dot{\Theta}^2) - \frac{mG}{r} = \text{constant} \tag{6-8}$$

and

$$r^2\dot{\theta} = H = \text{constant} \tag{6-9}$$

Now we want to express the various orbital parameters in terms of the launch elevation. A first step is to demonstrate that the angle α in Figure 6.1 is equal to 2E. As a means of proving this statement, write the equations for the circle and ellipse as

$$x^2+y^2 = R_e^2 \tag{6-10}$$

$$(x-ae)^2 + \frac{y^2}{1-e^2} = a^2 \tag{6-11}$$

Then compute the unit vectors tangent to the circle and ellipse at the launch point $(R_e\cos\theta_o, R_e\sin\theta_o)$:

$$i_c = i_x\sin\theta_o - i_y\cos\theta_o;\ \text{circle} \tag{6-12}$$

$$i_e = \frac{i_x\sin\theta_o - i_y(\cos\theta_o - e)}{\sqrt{1+e^2-2e\cos\theta_o}}\ ;\ \text{ellipse} \tag{6-13}$$

The elevation angle E is now determined as the inverse cosine of the dot product of the unit vectors i_c and i_e:

$$\cos E = i_c \cdot i_e = \frac{1-e\cos\theta_o}{\sqrt{1+e^2-2e\cos\theta_o}} \tag{6-14}$$

Use of the identity $\cos 2E = 2\cos^2 E-1$, yields

$$\cos 2E = \frac{1-e^2-2e\cos\Theta_o(1-e\cos\Theta_o)}{1+e^2-2e\cos\Theta_o} \tag{6-15}$$

Application of the Law of Cosines to the triangle formed by the vectors from the foci of the ellipse to the launch point (see Figure 6.1) yields:

$$4a^2e^2 = R_e^2 + (2a-R_e)^2-2R_e(2a-R_e)\cos\alpha \tag{6-16}$$

This relation can be rewritten in the form

$$\cos\alpha = \frac{1-e^2-2e\cos\Theta_o(1-e\cos\Theta_o)}{1+e^2-2e\cos\Theta_o} \tag{6-17}$$

when the definition

$$a = R_e(1-e\cos\Theta_o)/(1-e^2) \tag{6-18}$$

is used. (This is Eqn. 6-1 evaluated at the launch point $r = R_e$, $\Theta = \Theta_o$.) Inspection of the expressions for $\cos\alpha$ and $\cos 2E$ demonstrates that $\alpha = 2E$.

The Law of Sines can also be applied to the triangle to obtain an alternate expression for the major-axis of the ellipse:

$$\frac{2a-R_e}{\sin\Theta_o} = \frac{R_e}{\sin(\pi-2E-\Theta_o)} \tag{6-19}$$

or

$$a = \frac{R_e}{2}\left[1 + \frac{\sin\Theta_o}{\sin(2E+\Theta_o)}\right] \tag{6-20}$$

Also from this triangle we have that

$$\frac{2ae}{\sin 2E} = \frac{2a-R_e}{\sin\Theta_o} \tag{6-21}$$

The use of Eqn. (6-20) in Eqn. (6-21) yields, after some manipulation, the following relation for eccentricity:

$$e = \frac{\sin E}{\sin(E+\Theta_o)} \tag{6-22}$$

If the total energy at apogee is equated to the total energy at perigee, then it can be shown that the angular momentum, H, is given by

$$H = \sqrt{Ga(1-e^2)} \tag{6-23}$$

Now, the total energy at launch can be equated to the total energy at apogee ($r = a(1+e)$ at apogee):

$$\frac{1}{2}v_L{}^2 - \frac{G}{R_e} = \frac{1}{2}\,\frac{H^2}{a^2(1+e)^2} - \frac{G}{a(1+e)} = -\frac{G}{2a} \tag{6-24}$$

or,

$$v_L = \sqrt{\frac{G}{R_e}\left(\frac{2a-R_e}{a}\right)} \tag{6-25}$$

The use of Eqn. (6-20) for a gives the desired result:

$$v_L = \sqrt{\frac{2G}{R_e}\left[\frac{\sin\Theta_o}{\sin\Theta_o+\sin(2E+\Theta_o)}\right]} \tag{6-26}$$

The eccentric anomaly at launch (see Eqn. 6-2) is given by

$$E_{CL} = \cos^{-1}\left[\frac{a-R_e}{a\epsilon}\right] \tag{6-27}$$

and the eccentric anomaly at impact is

$$E_{CI} = 2\pi - E_{CL} \tag{6-28}$$

From E_{CL} and E_{CI}, the mean anomalies at launch and impact can be computed. The difference between the two is

$$\Delta M = M_I - M_L = 2(\pi - E_{CL} + e\sin E_{CL}) \tag{6-29}$$

Use of this value of ΔM, then yields an expression for time of flight

$$\Delta t = t_I - t_L = \Delta M/n = \sqrt{\frac{a^3}{G}}\,\Delta M \tag{6-30}$$

This completes the summary of the elliptic orbit parameters for object motion in a central force field.

6.2 Trajectory Initialization

The summary presented above in Section 6.1 gives the parameters of a Keplerian elliptic orbit in a convenient form for use in obtaining an initial estimate of the target state vector. In this Section, the equations will be specialized to accommodate particular trajectory constraints.

Let the launch point have longitude λ_L and geodetic latitude ϕ_L and let the impact point have values λ_I and ϕ_I. The geocentric latitude is readily computed from the geodetic latitude. A unit vector from the center of the earth to the launch point is given by

$$\imath_L = \imath_x \cos\phi_{cL} \cos\lambda_L + \imath_y \cos\phi_{cL} \sin\lambda_L + \imath_z \sin\phi_{cL} \tag{6-31}$$

During the flight of the target the earth rotates an amount $\omega_e t_f$, where ω_e is the rotation rate and t_f is the time of flight (initially unknown). The impact longitude is rotated by this amount

$$\lambda_{I'} = \lambda_I + \omega_e t_f \tag{6-32}$$

and a unit vector in the direction of this rotated impact point is

$$\imath_{I'} = \imath_x \cos\phi_{cI} \cos\lambda_{I'} + \imath_y \cos\phi_{cI} \sin\lambda_{I'} + \imath_z \sin\phi_{cI} \tag{6-33}$$

The earth central angle, $2\Theta_o$, between the launch and impact points is given by

$$\cos 2\Theta_o = \imath_{I'} \cdot \imath_L \tag{6-34}$$

or

$$\cos 2\Theta_o = \cos\phi_{cI} \cos\phi_{cL} \cos(\lambda_L - \lambda_{I'}) + \sin\phi_{cI} \sin\phi_{cL} \qquad (6\text{-}35)$$

It is useful to have a vector in the trajectory plane which is orthogonal to the launch unit vector:

$$n = (i_L \times i_{I'}) \times i_L = i_{I'} - \cos 2\Theta_o i_L \qquad (6\text{-}36)$$

This vector is tangent to the unit sphere and in a direction from launch toward impact. The magnitude of this vector is readily shown to be $|\sin(2\Theta_o)|$, so

$$i_n = \frac{n}{|n|} = (i_{I'} - \cos 2\Theta_o i_L)/\sin 2\Theta_o \qquad (6\text{-}37)$$

Since the trajectories are assumed to always be the short way around the earth, $0 < \Theta_o < \pi/2$ and the magnitude sign on $\sin 2\Theta_o$ is unnecessary.

The velocity vector at launch lies in the plane defined by the unit vectors i_L and i_n. If the launch speed is v_L and the launch elevation is E, then the launch velocity vector in the ECI frame can be written as

$$v_L = v_L(i_L \sin E + i_n \cos E) \qquad (6\text{-}38)$$

This velocity vector combined with a vector, $R_L i_L$, from the center of the earth to the launch point gives a complete ECI state vector at the launch point (R_L is the radius of the earth at the launch point). By means of the coordinate transformations defined in Section 5, this state vector can be put

into any desired coordinate system. The remaining part of the problem -- namely, assigning appropriate values to v_L and E--is considered below.

6.2.1 Minimum Energy Option

It is frequently desirable to fly the target along the trajectory which requires the minimum amount of energy at launch. This is achieved by minimizing the launch speed. The minimum launch speed in the ECI frame occurs for a launch elevation of

$$E = \frac{\pi}{4} - \frac{\Theta_o}{2} \tag{6-39}$$

Unfortunately, this is minimum energy in the ECI frame, rather than an earth surface fixed frame where the minimum likely will occur under different conditions due to coriolis effects. If the components of the launch position are denoted by x_L, y_L, z_L and the components of the i_n vector by n_x, n_y, n_z, then the velocity components in the ECF frame are

$$\begin{aligned} v_x &= v_L[\sin E(x_L/R_L) + \cos E\ n_x] + \omega_e y_L \\ v_y &= v_L[\sin E(y_L/R_L) + \cos E\ n_y] - \omega_e x_L \\ v_z &= v_L[\sin E(z_L/R_L) + \cos E\ n_z] \end{aligned} \tag{6-40}$$

The magnitude squared of the velocity in the ECF frame is

$$v^2 = v_L^2 + \omega_e^2(x_L^2+y_L^2) + 2\omega_e v_L \cos E(y_L n_x - x_L n_y) \tag{6-41}$$

This relation indicates that the elevation angle which minimizes v^2 will only coincide with the elevation which minimizes v_L^2 when the last term is zero. The last term contains the z-component of the cross product of $\imath_L$ and $\imath_n$ which is non-zero unless the trajectory plane (ECI) contains the North Pole.

The procedure employed for minimization of v^2 is to search for the elevation angle E which results in zero slope:

$$\frac{dv^2}{dE} = -2v_L \left\{ \frac{v_L \cos(2E+\Theta_o) + \beta \cos E \cos\Theta_o}{\sin\Theta_o + \sin(2E+\Theta_o)} \right\} \tag{6-42}$$

where

$$\beta = \omega_e(y_L n_x - x_L n_y)$$

Note that the minimum occurs at $E = \pi/4 - \Theta_o/2$ for a Polar orbit (β=0). An initial guess of $E = \pi/4 - \Theta_o/2$ is made, followed by steps of 0.01 radians until the zero of dv^2/dE is bracketed. Then a two-point linear approximation of dv^2/dE is used in an iterative scheme to locate the value of E which minimizes v^2. This value of E is then used to compute the orbit parameters a, e, E_{CL}, ΔM and Δt. The new time of flight Δt is compared to the previous value and the whole procedure repeated, unless the difference is less than one second.

Finally, the ECI state vector with the position vector, $R_L \imath_L$, and the velocity vector υ_L [given by Eqn. (6-38)] is transformed to an earth surface fixed XYZ frame (Radar XYZ). The final optimization of the state vector (which, among other things, accounts for flattening of the earth and for the effects of the second gravitational harmonic) is discussed in Section 6.3.

6.2.2 Launch Angle Specified

This option requires the determination of an initial velocity vector ν_L in the ECI frame which when transformed to an earth surface fixed frame (at the launch point) has the proper elevation of the velocity vector. That is, if v_x, v_y and v_z are the components of the velocity vector in the earth surface frame, then

$$E_F = \tan^{-1}(v_z/\sqrt{v_x^2+v_y^2}) \tag{6-43}$$

must be made equal to the desired launch angle, E_D.

An iterative procedure for the determination of the velocity vector ν_L follows. For a given value of the elevation E in the ECI frame, compute the corresponding launch speed (Eqn. 6-26), and launch velocity vector (Eqn. 6-38). Convert the ECI launch state vector thus determined to an earth surface XYZ frame at the launch point. Then compute the launch elevation angle from Eqn. (6-43) and compare it to the desired value E_D. If E_F is not close enough to E_D (e.g., 0.001 radians), choose a new value of E and repeat the calculations. The initial choice of E is taken to be E_D; thereafter, the new is derived from the old as

$$E_{n+1} = E_n + (E_D - E_{Fn})$$

Once the elevation E and launch speed v_L are determined, the orbit parameters a, e, E_{CL}, ΔM and Δt are computed. The new time of flight is

compared to the previous value and the whole procedure repeated, unless the difference is less than one second.

The final ECI state vector is then transformed to the launch point "Radar" XYZ frame. Launch speed, azimuth and elevation are computed from the Radar XYZ velocity vector to provide the initialization of the final trajectory optimization discussed below in Section 6.3.

6.2.3 Re-entry Angle Specified

This option is very similar to that described above for the launch angle specified case. In fact, the initialization procedure is exactly the same, except that the desired launch elevation is given by

$$E_D = |E_R| + 0.017$$

where E_R is the re-entry angle in radians ($E_R < 0$). The additional 0.017 radians (1 degree) is the result of observing that typically re-entry angles (magnitude) are smaller than the launch angles by 0.01 to 0.02 radians.

6.3 Launch State Vector Optimization

The procedures outlined in Section 6.2 above lead to reasonably good estimates of launch state vectors based on a homogeneous, spherical earth model. As a consequence of the simplified earth model, the target trajectory will not impact at exactly the desired coordinates. The problem then is to perturb the launch velocity vector in such a way that the target will impact

at the desired point and, in addition, will satisfy the launch angle, re-entry angle or minimum energy constraint.

An iterative technique for this launch state vector optimization is presented below in a general mathematical context. Then the specific cases of interest--launch angle specified, re-entry angle specified, minimum energy--are considered as three separate special examples.

6.3.1 A Solution of Nonlinear Equations

The impact longitude, impact latitude and re-entry angle are nonlinear functions of the launch velocity vector due to the oblateness of the earth and due to the gravitational harmonics of the earth's field. In this section, an iterative technique for perturbing the launch velocity vector, such that the trajectory will ultimately impact at the correct latitude and longitude with the desired re-entry angle (if specified), will be presented.

Let ν be the launch velocity vector and let y be a vector containing the quantities--impact longitude, impact latitude and re-entry angle. Actually, it is the vector, $q = y - y_d$, which is to be minimized. In this case, y_d is the vector of the desired values of y. The relation between q and ν can be expressed as

$$q = f(\nu) \tag{6-44}$$

where $f(\nu)$ is a nonlinear function of the vector ν. The objective is to determine ν such that $q = 0$. A perturbation $\Delta\nu$ of the vector ν results in

$$q + \Delta q = f(\nu+\Delta\nu) + J\Delta\nu \tag{6-45}$$

where J is the Jacobian

$$J = \begin{bmatrix} \frac{\partial q_1}{\partial v_1} & \frac{\partial q_1}{\partial v_2} & \frac{\partial q_1}{\partial v_3} \\ \frac{\partial q_2}{\partial v_1} & \frac{\partial q_2}{\partial v_2} & \frac{\partial q_2}{\partial v_3} \\ \frac{\partial q_3}{\partial v_1} & \frac{\partial q_3}{\partial v_2} & \frac{\partial q_3}{\partial v_3} \end{bmatrix} \tag{6-46}$$

Now let ν_o be an initial guess (obtained from the initialization described in Section 6.2). Then,

$$q_o = f(\nu_o) \tag{6-47}$$

If

$$q = f(\nu_o+\Delta\nu_o) = f(\nu_o) + J_o\Delta\nu_o$$

is to be zero, then it is required that

$$f(\nu_o) + J_o\Delta\nu_o = q_o+J_o\Delta\nu_o = 0$$

or,

$$\Delta\nu_o = -J_o^{-1}q_o = -H_oq_o \tag{6-48}$$

where $H \triangleq J^{-1}$. Now let

$$J_1 = J_o + A \tag{6-49}$$

be a new estimate of the Jacobian where A is a perturbation on the previous estimate. Then

$$q_1 = f(v_o + \Delta v_o) = q_o + J_1 \Delta v_o$$

$$= q_o + J_o \Delta v_o + A \Delta v_o$$

or

$$q_1 = A \Delta v_o, \tag{6-50}$$

since $J_o \Delta v_o = -J_o H_o q_o = -q_o$. The problem now is to determine the matrix A such that Eqn. (6-50) is satisfied. If it is further required that

$$A \Delta u = 0 \tag{6-51}$$

for Δu orthogonal to Δv_o; i.e.,

$$\Delta v_o^T \Delta u = 0, \tag{6-52}$$

then it can be shown that A has the unique solution

$$A_o = \frac{q_1 \Delta v_o^T}{\Delta v_o^T \Delta v_o} \tag{6-53}$$

This gives a new value for the Jacobian, $J_1 = J_o + A_o$, but what is actually required is the inverse of J_1:

$$H_1 = J_1^{-1} = \left[J_o + A_o\right]^{-1}$$

$$= \left[H_o^{-1} + \frac{q_1 \Delta v_o^T}{\Delta v_o^T \Delta v_o}\right]^{-1}$$

$$= \left[I + \frac{H_o q_1}{\Delta v_o^T \Delta v_o} \Delta v_o^T\right]^{-1} H_o \tag{6-54}$$

The well known matrix identity (Householder Identity)

$$(I + uv^T)^{-1} = I - \frac{uv^T}{1 + v^T u} \tag{6-55}$$

permits H_1 to be expressed as

$$H_1 = H_o - \frac{H_o q_1 \Delta v_o H_o}{\Delta v_o^T \Delta v_o + \Delta v_o^T H_o q_1} \tag{6-56}$$

Thus, when initial guesses of v_o and H_o, are available (or computed), the iterative procedure is to compute $q_o = f(v_o)$, $\Delta v_o = -H_o q_o$, H_1, $v_1 = v_o + \Delta v_o$, etc., until $|q_n|$ satisfies the desired error criterion. The initial value for the vector v_o is obtained by the procedures described in Section 6.2. Initialization of H_o is accomplished by perturbing v_o, one component at a time, and computing the changes in the components of q; i.e.,

$$J_{oij} = \frac{q_i - q_{oi}}{\Delta v_{oj}} \Bigg/ _{\Delta v_{ok} = 0,\ k \neq j} \tag{6-57}$$

The resultant matrix J_o is then inverted to obtain H_o.

The iterative technique for the solution of nonlinear equations, described above, is quite general. It will now be specialized to the cases of interest.

First note that the function $f(v)$ used to relate v and q is simply the ballistic equations of motion described in Section 4. The velocity vector, v,

$$v = \begin{bmatrix} \text{Launch speed} \\ \text{Launch azimuth} \\ \text{Launch elevation} \end{bmatrix} \tag{6-58}$$

is converted to earth surface fixed XYZ coordinates at the launch point. This is followed by a conversion to an ECI state vector for use by the trajectory integration routine. The trajectory is next integrated to either the re-entry altitude or the impact point (or both) and the appropriate values for the y vector are computed, where

$$y = \begin{bmatrix} \text{Re-entry angle} \\ \text{Impact longitude} \\ \text{Impact latitude} \end{bmatrix} \tag{6-59}$$

The q vector is simply the difference between the computed y vector and the desired vector y_d.

6.3.2 Launch Angle Specified

For this case, the launch elevation angle is held fixed at the desired value. Launch speed and launch azimuth in an earth surface (radar) fixed frame are perturbed as described above. When the iterative procedure has converged to the impact point (longitude and latitude), the launch velocity vector still has the desired elevation angle but the launch speed and launch azimuth are altered to achieve the proper impact point including the effects of gravitational harmonics and the oblateness of the earth ellipsoid.

The constraint on re-entry angle does not apply in this case, so the first column of the H matrix is set to zero to prevent perturbations of the velocity vector v due to re-entry angle errors. Also, the launch elevation angle is not included in the perturbations so the third row of the H matrix is set to zero. Thus,

$$H = \begin{bmatrix} 0 & H_{12} & H_{13} \\ 0 & H_{22} & H_{23} \\ 0 & 0 & 0 \end{bmatrix} \qquad (6\text{-}60)$$

It is readily shown that the method of updating the H matrix (Eqn. 6-56) results in a propagation of the null elements after they are initially zeroed.

Subroutine NONLIN (entry point SLV2) is called to perform the launch velocity optimization when the launch elevation angle is specified.

6.3.3 Re-entry Angle Specified

For this case, the re-entry angle enters the computations as a constraint. Also, the launch elevation angle is perturbed in the optimization. The *H* matrix used in generating launch velocity vector perturbations therefore has all non-zero elements. Convergence of the iterative procedure, in this case, results in a launch velocity vector which impacts at the desired longitude and latitude and which has the requested re-entry angle.

Subroutine NONLIN (entry point SLV3) is called to perform the launch velocity optimization when the re-entry angle is specified.

6.3.4 Minimum Energy Optimization

The procedure for launch velocity vector optimization subject to the minimum energy constraint is somewhat more complicated than the above two cases. It is first assumed that the initialization described in Section 6.2.1 is close to the true minimum. Then, the launch angle specified option is called three times with $E = E_{opt}$, $E = E_{opt}+\Delta E$ and $E = E_{opt}-\Delta E$, where E_{opt} is the launch elevation angle derived from the initialization described in Section 6.2.1 and ΔE is currently set to 0.005 radians. The three launch speed-launch elevation pairs are next fitted with a parabola to determine the elevation which minimizes the launch speed. A final call to Subroutine NONLIN (entry point SLV2) with this elevation angle results in a launch velocity vector corresponding to a minimum energy trajectory between the specified launch and impact points.

6.4 Trajectory Information

The presentation given above concerns the generation of target trajectories subject to certain constraints. There is the alternate possibility that a target state vector for a particular trajectory is available for input to the program directly. In this case, it is desirable to determine various parameters (e.g., launch point, impact point, re-entry angle) of the trajectory for documentation of a particular run of the ARIES Program. The mathematics for computation of the relevant trajectory parameters are presented below.

6.4.1 Input State Vectors

There are eight options for specification of a target trajectory as explained in Reference 1, Section 5.3. Options 6, 7 and 8 are the minimum energy, launch angle specified, and re-entry angle specified options which require the trajectory generation processing discussed above. The other five options require the input of a complete state vector in a particular coordinate frame as follows.

1. ECI XYZ
2. Radar XYZ
3. Radar RAE
4. Phased array XRF
5. Phased array RUV

Each of these five options requires a state vector time of validity (tag time). Options 2 through 5 also require the radar longitude, geodetic latitude and

height above the earth ellipsoid to be specified in the input stream. Options 4 and 5, in addition, require the azimuth and elevation angles of the phased array boresight. The ECI State Vector (Option 1) is referenced to a particular longitude at the launch time (TAL = 0); this longitude is a required input. (Option 9 for a satellite orbit has not been implemented.)

The first step in the handling of these input state vectors is to call a series of coordinate transformation subroutines to finally obtain a state vector in ECI coordinates referenced to the Greenwich meridian. This ECI State Vector is then integrated backward in time to the launch point (Subroutine LAUNCH) and forward in time to the altitude of re-entry and to the impact point (Subroutine IMPACT). The State Vector returned from Subroutine LAUNCH has a tag time referenced to time of launch (TAL).

6.4.2 Launch, Impact and Re-entry Parameters

Both the LAUNCH and the IMPACT Subroutines require trajectory integration to a specified altitude (zero for launch and impact points, 300 Kft. for re-entry). At the specified altitude, the state vector is appropriately transformed to determine the target longitude and geodetic latitude. The launch elevation angle, impact angle or re-entry angle is also determined as the angle between the velocity vector and the local horizontal plane. The azimuth angle of the velocity vector at launch is also determined. Each of these computations is performed in a radar XYZ frame centered at the longitude, latitude and altitude of the target.

Consider an ECI target state vector, x_I, somewhere along a trajectory. The altitude of the target (above the earth ellipsoid) is given approximately by (Eqn. 2-14):

$$h = r - \frac{R_n}{\sqrt{1-\varepsilon^2\cos^2 A}} \tag{6-61}$$

where $r = \sqrt{x_I^2+y_I^2+z_I^2}$, $\cos A = \rho/r$ and $\rho = \sqrt{x_I^2+y_I^2}$. (As before R_n is the North polar radius and ε is the eccentricity of the earth ellipsoid.) The angle A is approximately equal to the geocentric latitude of the point on the surface of the earth from which altitude is measured (i.e., the normal to the earth ellipsoid passing through the target position). A first approximation to the geodetic latitude is therefore

$$\phi = \tan^{-1}\left[\frac{z_I}{(1-\varepsilon^2)\rho}\right] \tag{6-62}$$

The longitude of the target is given by

$$\lambda = \tan^{-1}\left(\frac{y_I}{x_I}\right) - \omega_e\Delta t,\ 0 \le \lambda \le 2\pi \tag{6-63}$$

where the second term accounts for the rotation of the earth from the reference time of the ECI frame (usually TAL = 0) to the tag time of the state vector.

The parameters λ, ϕ and h give the position of the target with respect to the surface of the earth. It is also desirable to determine the azimuth and elevation of the velocity vector with respect to an earth surface fixed frame

at this position. To accomplish this, the following three unit vectors are defined:

$$i_1 = \cos\phi\left(\frac{x_I}{\rho}\, i_x + \frac{y_I}{\rho}\, i_y\right) + \sin\phi\; i_z \tag{6-64}$$

$$i_2 = -\frac{y_I}{\rho}\, i_x + \frac{x_I}{\rho}\, i_y \tag{6-65}$$

$$i_3 = -\sin\phi\left(\frac{x_I}{\rho}\, i_x + \frac{y_I}{\rho}\, i_y\right) + \cos\phi\; i_z \tag{6-66}$$

The unit vector i_1 is normal to the earth ellipsoid at latitude ϕ; i_2 is normal to i_1, parallel to the equatorial plane and points to the east; i_3 is normal to both i_1 and i_2 and points to the north. That is; i_1, i_2 and i_3 correspond to the Z, X and Y axes of a radar XYZ frame. The components of the velocity vector in the ECF frame are

$$\begin{aligned} \dot{x}_F &= \dot{x}_I + \omega_e y_I \\ \dot{y}_F &= \dot{y}_I - \omega_e x_I \\ \dot{z}_F &= \dot{z}_I \end{aligned} \tag{6-67}$$

The target speed in the ECF frame (or radar XYZ frame) is

$$v = \sqrt{\dot{x}_F^2 + \dot{y}_F^2 + \dot{z}_F^2} \tag{6-68}$$

The components of the velocity vector in the earth surface fixed frame are obtained by taking the following dot products

$$v_z = \imath_1 \cdot v_F = \cos\phi\left[\frac{x_I\dot{x}_I+y_I\dot{y}_I}{\rho}\right] + \sin\phi\ \dot{z}_I \tag{6-69}$$

$$v_x = \imath_2 \cdot v_F = \frac{-y_I\dot{x}_F+x_I\dot{y}_F}{\rho} \tag{6-70}$$

$$v_y = \imath_3 \cdot v_F = -\sin\phi\left[\frac{x_I\dot{x}_I+y_I\dot{y}_I}{\rho}\right] + \cos\phi\ \dot{z}_I \tag{6-71}$$

where $v_F = \imath_x\dot{x}_F+\imath_y\dot{y}_F+\imath_z\dot{z}_F$. Note that the effects of the rotation of the earth cancel in the expressions for v_z and v_y. The elevation angle of the velocity vector (which may be the launch angle, impact angle or re-entry angle) is given by

$$E_V = \sin^{-1}\left(\frac{v_z}{v}\right) \tag{6-72}$$

and the azimuth angle of the velocity vector is given by

$$A_V = \tan^{-1}\left(\frac{v_x}{v_y}\right) \tag{6-73}$$

The computation of λ, ϕ, h, E_V and A_V for an ECI state vector are performed in Subroutine ANGLES. Subroutines LAUNCH and IMPACT both call Subroutine ANGLES after the ECI state vector has been integrated forward or backward to the relevant target altitude. Various parameters of launch (λ,ϕ,E_V,A_V),

re-entry (λ,ϕ,E_V) and impact (λ,ϕ,E_V) are saved in the target information files.

Section 2 of the ARIES Test Report for a particular run of the Program (see Reference 1, Appendix II) contains trajectory information for each target. In particular, the following information is available in the Test Report:

1. Initial state vector data
2. Initial conditions for target tumbling and RCS computations
3. Launch parameters
4. Re-entry parameters
5. Impact parameters

SECTION 7--MAXIMUM LIKELIHOOD TRAJECTORY ESTIMATION

At present the only "tracking" algorithm built into ARIES is the Maximum-Likelihood Estimator (MLE). The MLE does not provide estimates of a target's position after each radar measurement, as do recursive-type tracking schemes such as the α-β tracker or a Kalman filter. Instead, the MLE provides ARIES only with an endpoint state vector estimate, valid at the end of the track interval(s) of interest. During the "track" interval itself, the MLE Subroutine MAXLIK simply stores up the (noisy, biased) radar position measurements, along with the variances of each measurement.

Once all such measurements have been accumulated, the MLE attempts to find that endpoint state vector estimate which minimizes the "residual errors" between the physical trajectory (uniquely defined by the endpoint state vector) and the set of radar measurements. The error used in this minimization is formed as the sum of the weighted, squared differences between the physical trajectory and the set of radar measurements. Each error residual is weighted according to the standard deviation of the corresponding position measurement.

Since each error residual is weighted according to the variance of the corresponding position measurement, the final state vector estimate corresponds to a minimum variance estimate. This minimum variance estimate is also a maximum likelihood estimate of the trajectory, when the measurement errors have a zero-mean, Gaussian distribution. If there are no bias errors in the data (such as those due to inadequate calibration or those due to imperfect correction of tropospheric or ionospheric refraction effects), the trajectory

estimation procedure described below corresponds to Maximum Likelihood Estimation. An extension of the procedure to include estimation of biases would again result in Maximum Likelihood Estimation of both the state vector and the biases.

7.1 The MLE Method

Let x_s be a reference ECI state vector at an arbitrary reference time t_s. (Note that x_s is the state vector to be perturbed to achieve the "optimal", weighted least-squares trajectory fit to the measurement data. For this study the state vector is valid at the endpoint of the tracking interval, but this is not a basic constraint.) Also let the set of radar measurements at N points in time $t_1, t_2, \ldots t_N$, be denoted by the vectors $r_m(t_i)$, $i=1, 2, \ldots N$. These radar position measurements may be either in RAE or RUV coordinates dependent on whether the radar is a dish type or a phased array type. The reference ECI state vector x_s is integrated successively to the times of the radar measurements. At each of these times, the state vector $x_s(t_i)$ is transformed to the appropriate radar frame (RAE or RUV) to form the set of "estimated" position vectors, $r_s(t_i)$. It is the weighted, squared differences between $r_m(t_i)$ and $r_s(t_i)$ which are minimized in the trajectory fitting process.

It has been shown in Section 5.2 that the transformation from an ECI state vector into radar XYZ (East-North-Up) coordinates can be performed via:

$$x_{sr}(t) = A(\phi,\lambda)G(t)x_s(t)+b(\phi) \tag{7-1}$$

where the transformation matrix $G(t)$ accounts for the rotation of the earth in time t (see Eqn. 5-1). The transformation matrix $A(\phi,\lambda)$ and the translation vector $b(\phi)$ are those used to transform from an ECF frame to a radar XYZ frame (see Eqns. 5-14 and 5.15). For a phased array radar, the additional transformation from radar XYZ to phased array XYZ (see Eqn. 5-26) must be included so that the vectors x_{sr} are defined in the proper coordinate system for the radars.

Also, we have denoted

$$x_s = \begin{bmatrix} x \\ y \\ z \end{bmatrix}_{ECI} \tag{7-2}$$

and

$$x_{sr} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}_{radar} \tag{7-3}$$

where the time dependence is implicit in these definitions.

From Equation (7-1) it is clear that a perturbation in x_s will lead to a corresponding perturbation in x_{sr}:

$$\delta x_{sr} = AG\ \delta x_s \tag{7-4}$$

Actually we want to relate perturbations in the reference state vector x_s to perturbations in the "estimated" target position r_s. To do this for the RAE

case, we use Eqn. (5-21) in Eqn. (5-20) to determine the relation between perturbations of r_s and x_{sr}:

$$\delta r_s = C\ \delta x_{sr} \tag{7-5}$$

where the matrix C is given by

$$C \triangleq \begin{bmatrix} \cos E \sin A & \cos E \cos A & \sin E \\ \cos A / R\cos E & -\sin A / R\cos E & 0 \\ -\sin E \sin A / R & -\sin E \cos A / R & \cos E / R \end{bmatrix} \tag{7-6}$$

For a phased array radar (RUV coordinates) the matrix C is

$$C \triangleq \frac{1}{R} \begin{bmatrix} RU & RV & RW \\ 1-U^2 & -UV & -UW \\ -UV & 1-V^2 & -VW \end{bmatrix} \tag{7-7}$$

where $W = \sqrt{1-U^2-V^2}$. Note that the matrix C varies as the target moves along its trajectory; for simplicity, this time dependence has not been explicitly included in the above relations.

Combining Equations (7-4) and (7-5) one can then relate perturbations in the position-measurement vector directly to perturbations in the ECI state vector:

$$\delta r_s = CAG\delta x_s \tag{7-8}$$

Back to the MLE problem: Given the reference state vector x_s, the MLE Subroutine MAXLIK first uses the equations of motion (Section 4.3) to obtain x_s

and then Equation (7-1) to obtain x_{sr} at each measurement time. Next the x_{sr} state vector is transformed into r_s (the set of target position estimates based upon the ECI reference vector x_s), using the appropriate transformations (given in Section 5.3 for an RAE radar, and in Sections 5.4 and 5.5 for an RUV radar).

In deriving the MLE estimator, we assume that perturbations in the reference state vector positions x_s and rates $\dot{x}_s$ at time t_s cause corresponding perturbations in $x_s(t_i)$ via the equation

$$\delta x_s(t_i) = \delta x_s + (t_i - t_s)\delta \dot{x}_s \tag{7-9}$$

This assumption becomes increasingly valid as δx_s and $\delta \dot{x}_s$ become small, as they do on successive MLE iterations.

The result of the above discussion is as follows: Given perturbations about the end-of-track ECI position estimate x_s and rate estimate $\dot{x}_s$ at a time t_s, we may determine the corresponding perturbation in the target position estimate at time t_i in radar measurement coordinates using Equations (7-8) and (7-9) to obtain:

$$\delta r_s(t_i, \delta x_s, \delta \dot{x}_s) = F(t_i)[\delta x_s + (t_i - t_s)\delta \dot{x}_s] \tag{7-10}$$

where we have defined

$$F(t_i) = C(t_i)A\, G(t_i) \tag{7-11}$$

At this point we have a set of radar measurement data $r_m(t_i)$, a set of "estimated" target positions $r_s(t_i)$ based on the reference ECI state vector $x_s(t_s)$

integrated to the appropriate times, and a set of position error estimates $\delta r_s(t_i)$ from Eqn. (7-10). These are now combined in a quadratic form as follows:

$$Q = \sum_{t=1}^{N} \{r_m(t_i) - r_s(t_i) - \delta r_s(t_i, \delta x_s, \delta \dot{x}_s)\}^T$$

$$W(t_i)\{r_m(t_i) - r_s(t_i) - \delta r_s(t_i, \delta x_s, \delta \dot{x}_s)\} \tag{7-12}$$

where W is a weighting matrix taking into account the quality of the radar's position measurements. Thus, for an RAE radar:

$$W(t_i) \triangleq \begin{bmatrix} \frac{1}{\sigma_R^2(t_i)} & 0 & 0 \\ 0 & \frac{1}{\sigma_A^2(t_i)} & 0 \\ 0 & 0 & \frac{1}{\sigma_E^2(t_i)} \end{bmatrix} \tag{7-13}$$

where $\sigma_R^2(t_i)$ is the range measurement variance, $\sigma_A^2(t_i)$ is the azimuth measurement variance, and $\sigma_E^2(t_i)$ is the elevation measurement variance for the radar measurements at time t_i. For a phased array radar (RUV data), σ_A^2 is replaced by σ_u^2 and σ_E^2 by σ_v^2. For simplicity, let the arguments t_i be simply denoted by the index i. Then with the additional definitions

$$\delta r(i) \triangleq r_m(i) - r_s(i), \tag{7-14}$$

and

$$\mathfrak{f}(i,\delta x_s,\delta \dot{x}_s) = \delta r(i) - \delta r_s(i,\delta x_s,\delta \dot{x}_s), \tag{7-15}$$

Eqn. (7-12) becomes:

$$Q = \sum_{i=1}^{N} \mathfrak{f}(i,\delta x_s,\delta \dot{x}_s)^T W(i)[\mathfrak{f}(i,\delta x_s,\delta \dot{x}_s)] \tag{7-16}$$

If the components of δx_s are denoted by δx_s, δy_s, δz_s, then to minimize Q one must set the partial derivatives of Q with respect to δx_s, δy_s, and δz_s to zero:

$$\frac{\partial Q}{\partial \alpha} = 2\sum_{i=1}^{N} [\mathfrak{f}(i,\delta x_s,\delta \dot{x}_s)]^T W(i) \frac{\partial \mathfrak{f}(i,\delta x_s,\delta \dot{x}_s)}{\partial \alpha} = 0 \tag{7-17}$$

where α takes on the values δx_s, δy_s and δz_s. Three simultaneous equations are generated from Eqn. (7-17) which may be combined into the single matrix equation

$$\sum_{i=1}^{N} [\mathfrak{f}(i,\delta x_s,\delta \dot{x}_s)]^T W(i) F(i) = (0,0,0) \tag{7-18}$$

Similarly, if the components of $\delta\dot{x}_s$, are denoted by $\delta\dot{x}_s$, $\delta\dot{y}_s$, $\delta\dot{z}_s$, and if the partial derivatives of Q with respect to $\delta\dot{x}_s$, $\delta\dot{y}_s$ and $\delta\dot{z}_s$ are set to zero, then the following matrix equation results:

$$\sum_{i=1}^{N} [\mathcal{G}(i,\delta x_s,\delta\dot{x}_s)]^T W(i) F(i)(t_i - t_s) = (0,0,0) \tag{7-19}$$

The ML Estimator simply has to solve these linear equations for δx_s and $\delta\dot{x}_s$; then the end-of-track state vector estimate can be updated:

$$x_{s,new} = x_{s,old} + \delta x_s \tag{7-20}$$

$$\dot{x}_{s,new} = \dot{x}_{s,old} + \delta\dot{x}_s \tag{7-21}$$

The matrix equations for the ML Estimator, Eqns. (7-18) and (7-19), can be written in a more compact form as follows. First take the transpose of these equations and substitute the expression for $\mathcal{G}(i,\delta x_s,\delta\dot{x}_s)$ from Eqn. (7-15). Then define the following vectors and matrices

$$d_1 = \sum_{i=1}^{N} F^T(i) W(i) \delta r(i) \tag{7-22}$$

$$d_2 = \sum_{i=1}^{N} (t_i - t_s) F^T(i) W(i) \delta r(i) \tag{7-23}$$

$$M_1 = \sum_{i=1}^{N} F^T(i) W(i) F(i) \tag{7-24}$$

$$M_2 = \sum_{i=1}^{N} (t_i - c_s) F^T(i) W(i) F(i) \tag{7-25}$$

$$M_3 = \sum_{i=1}^{N} (t_i - t_s)^2 F^T(i) W(i) F(i) \tag{7-26}$$

With these definitions, Eqns. (7-18) and (7-19) become

$$d_1 = M_1 \delta x_s + M_2 \delta \dot{x}_s \tag{7-27}$$

$$d_2 = M_2 \delta x_s + M_3 \delta \dot{x}_s \tag{7-28}$$

Equations (7-27) and (7-28) may be combined into a single (partitioned) matrix equation

$$\left[\begin{array}{c} d_1 \\ \hline d_2 \end{array}\right] = \left[\begin{array}{c|c} M_1 & M_2 \\ \hline M_2 & M_3 \end{array}\right] \left[\begin{array}{c} \delta x_s \\ \hline \delta \dot{x}_s \end{array}\right] \tag{7-29}$$

which is readily solved by Subroutine MAXLIK via a call to Subroutine SIMEQN. In Subroutine MAXLIK of ARIES, the corresponding FORTRAN arrays are defined as

$$A \triangleq \left[\begin{array}{c|c} M_1 & M_2 \\ \hline M_2 & M_3 \end{array}\right] \tag{7-30}$$

and

$$B \triangleq \left[\begin{array}{c} d_1 \\ \hline d_2 \end{array}\right] \tag{7-31}$$

These are not to be confused with the A and b which were defined above in Equation (7-1).

Note that the ML Estimator is an iterative procedure: An initial, reference state vector x_s is determined (equal to the true target state vector in ARIES), then perturbations δx_s and $\delta \dot{x}_s$ are determined by the above procedure. The new value of x_s from Eqns. (7-20) and (7-21) is then used to repeat the process. Since ARIES starts with an excellent value for x_s, the convergence of this process is very rapid; in fact, only two iterations are used.

7.2 Error Residuals

The residual errors between the target-position measurement for the radar $r_m(i)$ and the target-position prediction $r_s(i)$ at each time t_i (where $r_s(i)$ is computed from x_s, the final-iteration value of the MLE end-of-track state vector estimate) are computed via

$$\text{Mean square range residuals} = \sum_{i=1}^{NPTMAX} [\delta r_1(i)]^2 \qquad (7\text{-}32)$$

where NPTMAX is the number of measurements made during the Monte Carlo run, and where δr_1 is the first component of the residual error vector δr defined in Equation (7-14). Similar mean square residuals are computed for the other two components of δr. These mean square residuals are computed for each Monte Carlo run, and then averaged over all runs. The resultant mean-square-residuals are printed out in Section 11 of the ARIES Test Report.

SECTION 8--HANDOVER STATE VECTOR QUALITY

In ARIES, the MLE state vector estimate of the position and velocity of a target at the end of a tracking interval (or extrapolated ahead in time beyond the end of the tracking interval) will not, of course, correspond exactly to the true position and velocity of the target, due to the various noises and biases which were present in the radar measurements, as explained fully in Reference 2. The error vector at any time between the extrapolated state-vector estimate $\hat{x}$ and the true state vector x_{true} is defined as

$$e = \hat{x} - x_{true} \tag{8-1}$$

where both $\hat{x}$ and x_{true} are 6-component ECI state vectors.

The volume spanned by the error vector e is of interest to the command guided intercept problem, since it represents the uncertainty in the handover intercept position of the target. If the interceptor is being guided by a different radar than the one which tracked the target (i.e., a radar with a different set of biases, tracking at a different elevation with different amounts of refraction, etc.), then the error as defined in Equation (8-1) is of chief interest, as described below in Section 8.1. If, however, the same radar is tracking both the target and the interceptor, then the biases would tend to wash out, and the uncertainty after subtracting out the mean error is of more significance, as described in Section 8.2.

8.1 Correlation Matrices

On the mth Monte Carlo simulation run the ARIES Test Program computes a "correlation" matrix estimate via

$$\hat{C}_m \triangleq e_m e_m^T \tag{8-2}$$

where e_m is the error defined above in Equation (8-1) on the mth Monte Carlo run. These correlation matrix estimates are then averaged over all runs:

$$\hat{C} \triangleq \frac{1}{MC} \sum_{m=1}^{MC} \hat{C}_m \tag{8-3}$$

It is clear from Equation (8-2) that each of the correlation matrices $\hat{C}_m$ is symmetric; hence $\hat{C}$ is symmetric and only the "lower half" is printed out in Section 12 of the ARIES Test Report, as explained in Reference 1.

For the error ellipsoid calculations (see Section 3.3), it is necessary to diagonalize the "position" correlation matrix P by means of an orthogonal transformation (P is the upper left-hand quadrant of C; P is a 3 x 3 matrix). This diagonalization is only possible if the matrix P is positive definite; i.e., if the quadratic form $x^T P x$ is non-negative for all real values of the variables x_i, and is zero only if each of the x_i variables is zero. For the P matrix, this statement translates to the condition

$$\frac{1}{MC} \sum_{m=1}^{MC} (x^T e_m)^2 = \frac{1}{MC} \sum_{m=1}^{MC} (x_1 e_{1m} + x_2 e_{2m} + x_3 e_{3m})^2 \geq 0$$

which is obviously non-negative. If MC=1 or MC=2, it can be readily shown that there are an infinite number of non-zero values of the x_i's for which the above quadratic form is zero. This violates the conditions for P to be positive definite. For MC ≥ 3, it is not generally possible to make the quadratic form equal to zero with non-zero values for the x_i's. Exceptions are possible if the error vectors are linearly related; this is unlikely in the case at hand, particularly for large values of MC. The reason why the position correlation matrix is singular for one or two Monte Carlo runs can be simply explained as follows: a volume in space requires three non-parallel vectors for its definition (one error vector defines a point, two error vectors define a plane). Consequently, for the ARIES Program to generate meaningful error volumes based on position errors, it is necessary to perform a minimum of three Monte Carlo runs.

8.2 Covariance Matrices

If the mean error $\bar{e}$ (obtained by averaging over all Monte Carlo runs) is subtracted from the error, an estimated covariance matrix could be obtained on each Monte Carlo run via

$$\hat{\mathcal{D}}_m \triangleq (e_m - \bar{e})(e_m - \bar{e})^T \qquad (8\text{-}4)$$

and then averaged over all Monte Carlo runs to obtain

$$\hat{\mathcal{D}} \triangleq \frac{1}{MC} \sum_{m=1}^{MC} \hat{\mathcal{D}}_m \qquad (8\text{-}5)$$

In practice, what is actually done by ARIES is to first compute $\hat{C}$ via Equation (8-3), and next compute

$$\bar{e} = \frac{1}{MC} \sum_{m=1}^{MC} e_m \tag{8-6}$$

and finally to obtain $\hat{\mathcal{D}}$ using the relation

$$\hat{\mathcal{D}} = \hat{C} - \bar{e}\,\bar{e}^T \tag{8-7}$$

As with the correlation matrix $\hat{C}$ this covariance matrix $\hat{\mathcal{D}}$ is symmetric and thus only the lower half appears in Section 12 of the ARIES Test Report.

The "position" covariance matrix P (the upper left-hand 3 x 3 sub-matrix of $\hat{\mathcal{D}}$) is singular for MC < 4; the process of subtracting the means of the errors effectively reduces the dimensionality of the space by one. Consequently, for the ARIES program to generate meaningful error volumes based on the position error covariance matrix, it is necessary to perform a minimum of four Monte Carlo runs.

8.3 Handover Error Ellipsoid

The correlation and covariance matrices described in the preceding sections are for the errors as measured in the basic ARIES ECI coordinate system described earlier in Section 3.1. The drawback of such matrices is that all elements are non-zero - that is the X, Y, Z error components are correlated in this ECI frame.

To eliminate the correlations between the error vector components requires the definition of a new coordinate frame. Let a "position" correlation matrix P be defined as the upper left-hand quadrant of a correlation matrix $\hat{C}$ (or of a covariance matrix $\hat{\mathcal{D}}$):

$$\hat{C} = \left[\begin{array}{c|c} P & \\ \hline & \end{array}\right] \tag{8-8}$$

Then we want to determine a new coordinate system such that the ECI position error vector $e^T = (e_1, e_2, e_3)$ is related to the position error vector $(e')^T = (e_1', e_2', e_3')$ in the new coordinate system via the transformation matrix Q:

$$e' = Qe \tag{8-9}$$

where Q is the normalized modal matrix of P with the property that $Q^{-1} = Q^T$. The new "position" correlation matrix P' is then easily shown to be related to P via

$$P' = Q^T P Q \tag{8-10}$$

Given the symmetric, nonsingular matrix P, the problem is to find the matrix Q such that $Q^T P Q$ is a diagonal matrix. This is a standard problem (c f., Hildebrand, Reference 5), equivalent to determining the eigenvalues of the matrix P via setting the following determinant to zero:

$$|P - \lambda I| = 0 \tag{8-11}$$

where λ is an eigenvalue. For this case there are three such eigenvalues; they are the roots of the cubic equation

$$\begin{vmatrix} P_{11}-\lambda & P_{12} & P_{13} \\ P_{21} & P_{22}-\lambda & P_{23} \\ P_{31} & P_{32} & P_{33}-\lambda \end{vmatrix} = -\lambda^3-p\lambda^2-q\lambda-r = 0 \tag{8-12}$$

where

$$p = -(P_{11}+P_{22}+P_{33}) \tag{8-13}$$

$$q = P_{22}P_{33}+P_{33}P_{11}+P_{11}P_{22}-P_{23}^2-P_{13}^2-P_{12}^2 \tag{8-14}$$

and

$$r = P_{13}^2P_{22}+P_{12}^2P_{33}+P_{23}^2P_{11}-2P_{12}P_{23}P_{13}-P_{11}P_{22}P_{33} \tag{8-15}$$

The solution of (8-12) is well known. Define

$$a = q - p^2/3 \tag{8-16}$$

$$b = r - \frac{pq}{3} + \frac{2}{27}p^3 \tag{8-17}$$

For a positive definite matrix, compute

$$\phi = \cos^{-1}\left[-\frac{b}{2}\sqrt{\frac{-27}{a^3}}\right] \tag{8-18}$$

and obtain the eigenvalues as

$$\lambda_1 = \left[2\sqrt{-\frac{a}{3}}\cos\frac{\phi}{3}\right] - \frac{p}{3} \tag{8-19}$$

$$\lambda_3 = \left[2\sqrt{-\frac{a}{3}}\cos\left(\frac{\phi}{3} + \frac{2\pi}{3}\right)\right] - \frac{p}{3} \tag{8-20}$$

$$\lambda_3 = \left[2\sqrt{-\frac{a}{3}}\cos\left(\frac{\phi}{3} - \frac{2\pi}{3}\right)\right] - \frac{p}{3} \tag{8-21}$$

Since the matrix P is positive definite, the eigenvalues $(\lambda_1,\lambda_2,\lambda_3)$ are all real and positive. It then follows that P' has the desired diagonal property:

$$P' = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \tag{8-22}$$

The square roots of the diagonal elements of such diagonalized "position" correlation (or covariance) matrices can be thought of as the semi-axes of an ellipsoidal error volume - that is, the errors in this new coordinate frame are uncorrelated and have rms values $\sqrt{\lambda_1}$, $\sqrt{\lambda_2}$ and $\sqrt{\lambda_3}$. λ_1, λ_2 and λ_3 are also printed out in Section 12 of the ARIES Test Report.

8.4 Handover Error Sphere

While the coordinate transformation Q used in the preceding section rendered uncorrelated errors, the eigenvalues λ_1, λ_2 and λ_3 alone are insufficient to specify the uncertainty volume of the target state vector at hand over. The orientation of the ellipsoid axes must also be taken into account. In order to avoid this complexity, it is often more useful to think of an "uncertainty sphere" - namely a sphere of radius R, centered at the

(extrapolated) handover point, with R chosen such that the error vector lies within the sphere with probability = 0.99.

The diagonalization of the position correlation (or covariance) matrix assures us that the error components in the new coordinate frame (see preceding section) are uncorrelated. If further we assume these errors to have "normal" or "Gaussian" distributions, then the probability of the error vector e' [defined in Equation (8-9)] is given by

$$p(e') = \frac{e^{-\frac{X^2}{2\lambda_1} - \frac{Y^2}{2\lambda_2} - \frac{Z^2}{2\lambda_3}}}{(2\pi)^{1.5}\sqrt{\lambda_1 \lambda_2 \lambda_3}} \triangleq p(X,Y,Z) \qquad (8\text{-}23)$$

(where for notational convenience the components e_1', e_2', e_3' of the error vector e' have been replaced by X, Y and Z, respectively). The probability of the error vector falling within a sphere of radius R is

$$p(|e'| < R) = 8\int_0^R dX \int_0^{\sqrt{R^2-X^2}} dY \int_0^{\sqrt{R^2-X^2-Y^2}} p(X,Y,Z)dZ \qquad (8\text{-}24)$$

where due to the symmetry we only have to integrate over one-eighth of the sphere. If we use Equation (8-23) in (8-24), along with a set of normalized variables defined by

$$x = \frac{X}{\sqrt{2\lambda_1}} \qquad (8\text{-}25)$$

$$y = \frac{Y}{\sqrt{2\lambda_2}} \tag{8-26}$$

$$z = \frac{Z}{\sqrt{2\lambda_3}} \tag{8-27}$$

then we obtain

$$p(|e'| < R) = \frac{8}{\pi^{1.5}} \int_0^{C_x} dx\, e^{-x^2} \int_0^{C_y\sqrt{1 - \frac{x^2}{C_x^2}}} dy\, e^{-y^2} \int_0^{C_z\sqrt{1 - \frac{x^2}{C_x^2} - \frac{y^2}{C_y^2}}} e^{-z^2}\, dz \tag{8-28}$$

where we have defined:

$$C_x = R/\sqrt{2\lambda_1} \tag{8-29}$$

$$C_y = R/\sqrt{2\lambda_2} \tag{8-30}$$

$$C_z = R/\sqrt{2\lambda_3} \tag{8-31}$$

Note that the integration is now over an ellipsoid rather than a sphere.

If we transform y and z to polar coordinates r and θ, where

$$r = \sqrt{y^2+z^2} \tag{8-32}$$

$$\theta = \tan^{-1}(z/y) \tag{8-33}$$

then we may replace the two inner integrals of Equation (8-28) with

$$\int_0^{\pi/2} d\Theta \int_0^{r(\Theta)} re^{-r^2}\, dr = -\frac{1}{2}\int_0^{\pi/2}\left[e^{-r^2(\Theta)}-1\right]d\Theta \tag{8-34}$$

This integration gives the area of an ellipse in the y-z plane. The semi-axes of the ellipse are $\sqrt{1-x^2/C_x^2}\, C_y$ and $\sqrt{1-x^2/C_x^2}\, C_z$. Also, the distance from the origin (y=z=0) to the ellipse is

$$r(\Theta) \triangleq \frac{\sqrt{1-\frac{x^2}{C_x^2}}\, C_y}{\sqrt{1-\left(1-\frac{C_y^2}{C_z^2}\right)\sin^2\Theta}} \tag{8-35}$$

Use of the double angle formula, $2\sin^2\Theta = 1-\cos2\Theta$, in Eqn. (8-35) results in the following relation

$$r^2(\Theta) = \frac{\left(1-\frac{x^2}{C_x^2}\right)C_y^2}{\frac{1}{2}\left(1+\frac{C_y^2}{C_z^2}\right)+\frac{1}{2}\left(1-\frac{C_y^2}{C_z^2}\right)\cos2\Theta} \tag{8-36}$$

or,

$$r^2(\Theta) = \frac{1}{k_1+k_2\cos2\Theta} \tag{8-37}$$

where

$$k_1 \triangleq \frac{\left(1+\frac{C_y^2}{C_z^2}\right)}{2\left(1-\frac{x^2}{C_x^2}\right)C_y^2} \tag{8-38}$$

and

$$k_2 \triangleq \frac{1 - \frac{C_y^2}{C_z^2}}{2\left(1 - \frac{x^2}{C_x^2}\right)C_y^2} \tag{8-39}$$

Substitution of Eqn. (8-37) into Eqn. (8-34) gives

$$\int_0^{\pi/2} d\Theta \int^{r(\Theta)} re^{-r^2}dr = \frac{1}{4}\left[\pi - \int_0^{\pi} \exp\left\{\frac{-1}{k_1+k_2\cos\phi}\right\}d\phi\right] \tag{8-40}$$

where the change of variables $2\Theta = \phi$ has been used.

When Equation (8-40) is used to replace the inner two integrals of (8-28), we obtain

$$p(|e'| < R) = \frac{2}{\pi^{1.5}}\int_0^{C_x} dx\ e^{-x^2}\left\{\pi - \int_0^{\pi}\exp\left\{\frac{-1}{k_1+k_2\cos\phi}\right\}d\phi\right\} \tag{8-41}$$

The double integral in Eqn. (8-41) cannot be reduced any further; therefore, it is evaluated numerically by a Gauss quadrature method.

If we specify an error sphere radius R, Equation (8-41) allows the computation of the probability that the error vector lies within the sphere. However, it is usually desired to find the value of R which gives a probability of 0.99. This value of R is found numerically by means of the secant iterative method described in Reference 4.

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